Math. 152, Sections 501–503 (Fulling) 17 December 2002

Final Examination – Solutions

Test Forms A and B were the same except for the order of the multiple-choice responses. This key is based on Form A.

Name: ______________________________ Section: ______________

POSSIBLY USEFUL FORMULAS

$$\sin^2 A = \frac{1-\cos(2A)}{2} \qquad \int \tan\theta\, d\theta = -\ln|\cos\theta| + C$$

$$\cos^2 A = \frac{1+\cos(2A)}{2} \qquad \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\sin A \sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B)$$
$$\sin A \cos B = \tfrac{1}{2}\sin(A-B) + \tfrac{1}{2}\sin(A+B)$$
$$\cos A \cos B = \tfrac{1}{2}\cos(A-B) + \tfrac{1}{2}\cos(A+B)$$

$$|E_M| \le \frac{K(b-a)^3}{24n^2} \quad \text{if } |f''(x)| \le K \text{ for all } x \text{ in } [a,b].$$
$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{if } |f''(x)| \le K \text{ for all } x \text{ in } [a,b].$$
$$|E_S| \le \frac{K(b-a)^5}{180n^4} \quad \text{if } |f^{(4)}(x)| \le K \text{ for all } x \text{ in } [a,b].$$

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1},$$

where M is an upper bound on $|f^{(n+1)}(c)|$ for all c in the interval concerned.

Part I: Multiple Choice (3 points each)

There is no partial credit. Do not use a calculator for symbolic operations, such as evaluating integrals and limits or attempting to sum infinite series.

1. A tank contains 5 kg of salt dissolved in 1000 L of water. To clean out the salt, fresh water is pumped into the tank at a rate of 25 L/min, the contents are well mixed, and the liquid is drained from the tank at the same rate. What are the DIFFERENTIAL equation and initial condition satisfied by the amount of salt in the tank?

   (A) $\dfrac{dy}{dt} = e^{-40t}\,, \quad y(0) = 25$

   (B) $\dfrac{dy}{dt} = -25\,, \quad y(0) = 5$

   (C) $\dfrac{dy}{dt} = -\dfrac{y}{40}\,, \quad y(0) = 5 \;\Leftarrow$ correct

   (D) $\dfrac{dy}{dt} = \dfrac{y}{40}\,, \quad y(0) = 1000$

   (E) $\dfrac{dy}{dt} = e^{t/40}\,, \quad y(0) = 1000$

   The concentration of salt is

   $$c(t) = y(t)/1000\,,$$

   so

   $$\frac{dy}{dt} = -25c = -\frac{y}{40}\,.$$

2. The theoretical upper bound on the error in the approximation $e^x \approx 1 + x$ for x in the interval $[-2, 2]$ is

   (A) $2e^2 \;\Leftarrow$ correct

   (B) $\dfrac{e}{2}$

   (C) e

   (D) $\dfrac{e^2}{3}$

   (E) 2

   In the notation of the cover sheet,

   $$n = 1\,, \quad a = 0\,, \quad f^{(n)}(x) = e^x\,.$$

   The maximum value occurs at $x = 2$, so $M = e^2$.

   $$|R_n| \le \frac{e^2}{2!}\,2^2 = 2e^2\,.$$

3. Calculate the vector cross product, $\vec{v}\times\vec{w}$, of the vectors $\vec{v}=\langle 2,1,2\rangle$ and $\vec{w}=\langle -1,1,1\rangle$.

(A) $\langle 1,4,-3\rangle$

(B) $\langle 1,-4,-3\rangle$

(C) $\langle -1,4,-3\rangle$

(D) $\langle -1,4,3\rangle$

(E) $\langle -1,-4,3\rangle$ $\Leftarrow$ correct

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} = \hat{\imath}\,(-1) + \hat{\jmath}\,(-4) + \hat{k}\,(3)$$

4. The angle between the vectors $\vec{v}$ and $\vec{w}$ in the previous problem is

(A) $\cos^{-1} 3^{-3/2}$ $\Leftarrow$ correct

(B) $\tan^{-1}\sqrt{26}$

(C) $\sin^{-1} 3^{-3/2}$

(D) $\cos^{-1}(26)^{-1/2}$

(E) $\frac{1}{2}\sqrt{26}$

$$|\vec{v}| = \sqrt{4+1+4} = 3\,, \quad |\vec{w}| = \sqrt{3}\,.$$

$$\vec{v}\cdot\vec{w} = -2+1+2 = 1\,.$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{|\vec{v}||\vec{w}|} = \frac{1}{3\sqrt{3}}\,.$$

5. $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{6} + \cdots$ is the Maclaurin series of

(A) e^x

(B) $\cos x$

(C) $\sin x$ $\Leftarrow$ correct

(D) $\ln x$

(E) $\ln(x-1)$

6. An ideal spring with a natural length of 3 feet is stretched to a length of 5 feet by a force of 2 pounds. How much work is done in stretching this spring to a length of 6 feet?

(A) 2 foot-pounds

(B) 9 foot-pounds $\Leftarrow$ correct

(C) 6 foot-pounds

(D) 3 foot-pounds

(E) 12 foot-pounds

The force law is $|F(x)| = kx$, where x is the amount of stretching. So $2 = k \cdot 1$, or $k = 2$. Now the work is

$$\int_0^{6-3} kx\,dx = \frac{1}{2}\,2x^2\Big|_0^3 = 9\,.$$

7. Suppose that $\left|\dfrac{a_{n+1}}{a_n}\right| \to L$ as $n \to \infty$. Then the series $\displaystyle\sum_{n=1}^{\infty} a_n$

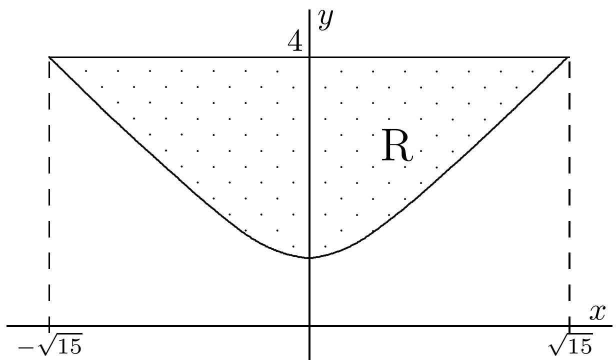
(A) converges only if $L = 0$.

(B) converges to L.

(C) converges to L only if $L \leq 1$.

(D) converges if $L < 1$. $\Leftarrow$ correct

(E) diverges unless $L < 1$.



Let's see how many things we can do with a hyperbola. More precisely, let R be the plane region bounded above by the line $y = 4$ and below by the hyperbolic segment

$$y = \sqrt{1+x^2}\,, \qquad -\sqrt{15} < x < \sqrt{15}\,.$$

The next six problems will be based on this geometry. (When it matters, all dimensions are in feet.)

8. The area of R can be calculated EXACTLY by

(A) Simpson's rule

(B) the trapezoidal rule

(C) both

(D) neither $\Leftarrow$ correct

The integrand $\sqrt{1+x^2}$ is not a quadratic (much less linear) polynomial. (In other words, K in the error formulas is not 0.)

9. The arc length of the hyperbolic segment is

(A) $\displaystyle\int_{-\sqrt{15}}^{\sqrt{15}} \left[1 + \frac{x}{\sqrt{1+x^2}}\right] dx$

(B) $\displaystyle 2\int_0^{\sqrt{15}} \sqrt{\frac{1+2x^2}{1+x^2}}\, dx$ $\Leftarrow$ correct

(C) $\displaystyle 2\int_0^{\sqrt{15}} \sqrt{2+x^2}\, dx$

(D) $\displaystyle 2\int_1^4 \sqrt{1+y^2}\, dy$

(E) $\displaystyle\int_1^4 [1+\sqrt{1-y^2}]\, dy$

$$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}\,;$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1+x^2} = \frac{1+2x^2}{1+x^2}\,.$$

10. A water tank 7 feet long has R as vertical cross section. If the tank is full of water, the work required to pump the water out over the top is

(A) $62.5\displaystyle\int_1^4 14\sqrt{y^2-1}(4-y)\,dy$
$\Leftarrow$ correct

Integrand is

$$\text{density} \times \text{length} \times \text{width} \times \text{depth}\,.$$

(B) $\displaystyle\int_1^4 7\sqrt{y^2-1}(4-y)\,dy$

(C) $62.5\displaystyle\int_1^3 7\sqrt{y^2-1}\,y\,dy$

(D) $\displaystyle\int_0^3 14\sqrt{y^2-1}\,y\,dy$

(E) $62.5\displaystyle\int_1^3 \sqrt{y^2-1}(3-y)\,dy$

11. When the tank is full of water, the hydrostatic force against one of the hyperbolic ends is

(A) the same as the answer to the previous question.

(B) 7 times the answer to the previous question.

(C) the answer to the previous question divided by 7. $\Leftarrow$ correct

(D) 62.5 times the answer to the previous question.

(E) the answer to the previous question divided by 62.5.

Integrand is

$$\text{density} \times \text{width} \times \text{depth}\,.$$

The pressure is independent of the length.

12. When R is revolved about the horizontal axis, the volume of the resulting solid region is

(A) 1

(B) $2\pi\sqrt{15}$

(C) 30π

(D) $\dfrac{15\pi}{2}$

(E) $\dfrac{4\pi}{3}(15)^{3/2}$ $\Leftarrow$ correct

Washer method:

$$\int_{-\sqrt{15}}^{\sqrt{15}} \pi(4^2 - y^2)\,dx = \int_{-\sqrt{15}}^{\sqrt{15}} \pi(15 - x^2)\,dx\,.$$

Cylinder method:

$$\int_1^4 2\pi y \cdot 2\sqrt{y^2-1}\,dy = 2\pi \int_0^{15} \sqrt{u}\,du\,.$$

13. If A is the area of R, the y coordinate of the centroid of R is

(A) the answer to the previous question divided by A.

(B) the answer to the previous question times $2\pi A$.

(C) the answer to the previous question divided by $2\pi A$. $\Leftarrow$ correct

(D) the answer to the previous question times $2\pi/A$.

(E) the answer to the previous question times A.

Method 1:

$$A\overline{y} = \int_1^4 (2x)y\,dy = 2\int_1^4 y\sqrt{y^2-1}\,dy$$

which is $1/2\pi$ times the cylinder integral above.

Method 2:

$$A\overline{y} = \int_{-\sqrt{15}}^{\sqrt{15}} \frac{1}{2}\,(4^2 - y^2)\,dx$$

which is $1/2\pi$ times the washer integral above.

Method 3: This fact is an instance of the Theorem of Pappus, stated on page 559 of Stewart.

14. $\displaystyle\int_0^{\pi/2} \cos(2x)\cos(3x)\,dx =$

(A) π

(B) $3/5 \ \Leftarrow$ correct

(C) 0

(D) 1

(E) $\pi/2$

From formula on cover page, this is

$$\int \frac{1}{2}[\cos(-x)+\cos(5x)]\,dx$$

which is elementary.

15. $\displaystyle\int_1^{\infty} \frac{dx}{x^2-2x+5} =$

(A) ∞

(B) π

(C) $\dfrac{3\pi}{2}$

(D) $\dfrac{\pi}{16}$

(E) $\dfrac{\pi}{4} \ \Leftarrow$ correct

Let $u = x-1$, so $u^2+4 = x^2-2x+5$. When $x = 1$, $u = 0$. So the integral is

$$\int_0^{\infty} \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2}\Big|_0^{\infty} = \frac{\pi}{4} - 0\,.$$

16. If the Maclaurin series of $g(x)$ is $2 - x + 3x^2 + \cdots$, then the series of $e^{-x^2}g(x)$ is

(A) $2 - 3x^2 + \cdots$

(B) $2 - x - 3x^2 + \cdots$

(C) $2 - 3x^4 + \cdots$

(D) $2 - x + x^2 + \cdots \ \Leftarrow$ correct

(E) impossible to determine from the information given.

$$[2 - x + 3x^2 + \cdots][1 - x^2 + \cdots] =$$

$$\begin{aligned} &2 - x - 3x^2 + \cdots \\ & - 2x^2 + \cdots \\ = {}&2 - x + x^2 + \cdots \end{aligned}$$

Part II: Write Out (10 points each except as indicated)

Give complete solutions ("show work"). Appropriate partial credit will be given. Do not use a calculator for symbolic operations, such as evaluating integrals and limits or attempting to sum infinite series.

17. *(12 points)* Determine whether each of these series or improper integrals converges or diverges. Say why: Be sure to name or quote the test(s) you use and check out the requirements of the test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{n}$ CIRCLE ONE: Converges <u>Diverges</u>

Explain: Harmonic series (p-test with $p = 1 \leq 1$).

(b) $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$ CIRCLE ONE: <u>Converges</u> Diverges

Explain: p-test (for integrals improper at 0) with $p = \frac{1}{2} < 1$. Explicitly:

$$2x^{1/2}\Big|_T^1 = 2 - 2T^{1/2} \to 2 < \infty \quad \text{as } T \to 0\,.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{7^{n+1}}{9^n}$ CIRCLE ONE: <u>Converges</u> Diverges

Explain: Geometric series $7\sum_{n=0}^{\infty}\left(\frac{7}{9}\right)^n$ with $r = \frac{7}{9} < 1$. Or, from first principles, use the ratio test:

$$\frac{7^{n+2}}{9^{n+1}}\,\frac{9^n}{7^{n+1}} = \frac{7}{9} \to \frac{7}{9} < 1\,.$$

18. Find the Maclaurin series of $f(x) = \tan^{-1} x$. *Hint:* Start from $(1 + x^2)^{-1}$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n \quad \text{(by substitution in the geometric series).}$$

$$
\begin{aligned}
\tan^{-1} x &= \int_0^x \frac{1}{1+t^2}\, dt \\
&= \sum_{n=0}^{\infty} \int_0^x (-1)^n t^{2n}\, dt \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} .
\end{aligned}
$$

19. A plane passes through the points $P\ (1,1,2)$, $Q\ (-1,0,3)$, and $R\ (2,1,1)$.

(a) Find an equation for this plane of the form $Ax + By + Cz = D$.

Two vectors parallel to the plane are

$$\overrightarrow{QP} = \langle 2,1,-1\rangle, \quad \overrightarrow{QR} = \langle 3,1,-2\rangle .$$

Therefore, a vector normal (perpendicular, orthogonal) to the plane is their cross product,

$$\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -1 \\ 3 & 1 & -2 \end{vmatrix} = \langle -1, 1, -1\rangle .$$

The equation of the plane is

$$0 = \vec{N} \cdot (\vec{r} - Q) = -1(x+1) + 1(y-0) - 1(z-3) ,$$

which simplifies to $\quad x - y + z = 2$.

(b) Find a parametric equation for this plane.

We can recycle the two parallel vectors from part (a).

$$\vec{r} = Q + s\ \overrightarrow{QP} + t\ \overrightarrow{QR} = (-1,0,3) + s\langle 2,1,-1\rangle + t\langle 3,1,-2\rangle ,$$

or

$$x = -1 + 2s + 3t , \quad y = s + t , \quad z = 3 - s - 2t .$$

(Many different but equivalent answers are possible.)

20. Find the solution of the differential equation $\dfrac{dy}{dx} + \dfrac{2}{x}\,y = \dfrac{\sin x}{x}$ satisfying $y(\pi) = 0$.

The integrating factor $I(x)$ is the reciprocal of the simplest nonzero solution of the associated homogeneous equation,

$$\frac{dy_{\mathrm{h}}}{dx} = -\frac{2y_{\mathrm{h}}}{x}\,;$$

$$\int \frac{dy_{\mathrm{h}}}{y_{\mathrm{h}}} = -2\int \frac{dx}{x}\,;$$

$$\ln y_{\mathrm{h}} = -2\ln x\,;$$

$$y_{\mathrm{h}} = x^{-2}\,;$$

$$I(x) = x^2\,.$$

Multiply the nonhomogeneous equation by I:

$$\frac{d}{dx}(x^2y) = x^2\,\frac{dy}{dx} + 2xy = x\sin x.$$

$$\begin{aligned} x^2y &= \int x\sin x\,dx \\ &\equiv \int u\,dv = uv - \int v\,du \\ &= -x\cos x + \int \cos x\,dx \\ &= -x\cos x + \sin x + C\,. \end{aligned}$$

Therefore,

$$y(x) = -\frac{\cos x}{x} + \frac{\sin x}{x^2} + \frac{C}{x^2}\,.$$

Apply the initial condition to find C:

$$0 = -\frac{\cos\pi}{\pi} + \frac{\sin\pi}{\pi^2} + \frac{C}{\pi^2} = \frac{1}{\pi} + \frac{C}{\pi^2}\,.$$

Thus $C = -\pi$.

21. Find the indefinite integral $\int (1-x^2)^{3/2}\,dx\,.$

Let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$, $(1-x^2)^{1/2} = \cos\theta$. Then

$$
\begin{aligned}
I &= \int \cos^4\theta\,d\theta \\
&= \int \left(\frac{1+\cos(2\theta)}{2}\right)^2\,d\theta \\
&= \frac{1}{4}\int [1+2\cos(2\theta)+\cos^2(2\theta)]\,d\theta \\
&= \frac{1}{4}\left[1+2\cos(2\theta)+\frac{1+\cos(4\theta)}{2}\right]\,d\theta \\
&= \int \left[\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right]\,d\theta \\
&= \frac{3}{8}\,\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta) \\
&= \frac{3}{8}\,\theta+\frac{1}{2}\sin\theta\cos\theta+\frac{1}{16}\sin(2\theta)\cos(2\theta) \\
&= \frac{3}{8}\,\theta+\frac{1}{2}\sin\theta\cos\theta+\frac{1}{8}\sin\theta\cos\theta(1-2\sin^2\theta) \\
&= \frac{3}{8}\sin^{-1}x+\frac{1}{2}\,x\sqrt{1-x^2}+\frac{1}{8}\,x\sqrt{1-x^2}(1-2x^2)\,,
\end{aligned}
$$

or, finally,

$$\int (1-x^2)^{3/2}\,dx = \frac{3}{8}\sin^{-1}x+\frac{5}{8}\,x\sqrt{1-x^2}-\frac{1}{4}\,x^3\sqrt{1-x^2}+C\,.$$

(There are other, equivalent, formulas that look slightly different. For example, parts of the square-root terms could be combined to make a term proportional to $x(1-x^2)^{3/2}$.)