Maple Lab, Week 26

The Pendulum Equation and Maple's `dsolve` Command

Background: The second order differential equation

$$\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{l}\sin(\theta) = 0 \tag{1}$$

is a mathematical model for the motion of a pendulum. The pendulum consists of a rigid, weightless rod of length l, pinned at one end, and with a mass m at the other end. The rod pivots about the pinned end, O, and is assumed to move in a single plane. The angular position of the rod at time t, measured counterclockwise from vertically downward, is $\theta = \theta(t)$. The term $c\frac{d\theta}{dt}$ accounts for the drag on the mass due to air resistance. You will be asked to derive the equation (1) in the exercises.

If we simplify (1) by assuming $\frac{g}{l} = 1$ and $c = 0$, we get

$$\frac{d^2\theta}{dt^2} + \sin(\theta) = 0. \tag{2}$$

Maple has a command, `dsolve`, for solving differential equations. The following example illustrates the use of `dsolve` to solve the simple initial value problem $y' + 3y = 0,\ y(0) = 2$. Additional examples may be found in the online help.

```
> de1 := diff(y(t),t) + 3*y(t) = 0;
> init1 := y(0) = 2;
> sol1 := dsolve({de1,init1},y(t));
```

Note that the solution is an equation. If we would like to plot the solution, for example, we could use the `plot` command together with the `rhs` command (for `r`ight `h`and `s`ide):

```
> ysol1 := rhs(sol1);
> plot(ysol1, t=0..1);
```

Sometimes `dsolve` can't solve a differential equation. For example, the equation (2) is nonlinear and happens to be too difficult for Maple, as the following commands illustrate:

```
> de2 := diff(th(t),t,t) + sin(th(t)) = 0;
> init2 := th(0) = 0, D(th)(0) = 2;
> sol2 := dsolve({de2,init2},th(t));
```

There are at least two things which can be done to remedy the situation: simplify the differential equation, or seek an approximate (i.e., numerical) solution.

As an example of the first approach, we can approximate $\sin(\theta)$ by the linear approximation $L(\theta) = \theta$ near $\theta = 0$. This gives the equation $\theta'' + \theta = 0$, which is easy for Maple to solve (and easy to solve by hand, as well):

```
> de3 := diff(th(t),t,t) + th(t) = 0;
> sol3 := dsolve({de3,init2},th(t));
```

To obtain a numerical solution, include `type = numeric`, or simply `numeric`, in the calling sequence:

```
> sol2 := dsolve({de2,init2},th(t),numeric);
```

The solution is a procedure which can be evaluated at any value of the independent variable:

```
> sol2(0); sol2(1);
```

One way to plot the solution is to create a list of ordered pairs and use the `plot` command:

```
> plt := NULL:
> for i from 0 by .1 to 10 do
> plt := plt, [rhs(sol2(i)[1]), rhs(sol2(i)[2])]:  od:  plot([plt]);
```

Notice that $\theta(t) \to \pi$ as t increases. What motion of the pendulum is represented by this solution?

Exercises: All graphs in the following exercises are to be produced using the `plot` command, as in the examples above.

1. Derive the equation (1), when there is no drag, i.e., $c = 0$. [Hint: Recall *torque* = *I·alpha*.

2. Display graphs of the solutions `sol2` and `sol3` on the same coordinate system, for $0 \le t \le 10$. [Hint: Recall the `display` command in the `plots` package.]

3. Repeat Exercise 2, but for the initial condition $\theta(0) = 0$, $\theta'(0) = .5$, on the interval $0 \le t \le 40$. Identify which graph is which. [Hint: How long does it take the solution to $\theta'' + \theta = 0$ to complete six periods?]

4. Use conservation of energy to explain why the initial condition $\theta(0) = 0$, $\theta'(0) = 2$ corresponds to the solution of (2) for which the pendulum converges to the vertically upright position, as $t \to \infty$.

Do either Exercise 5 or Exercise 6; extra credit for doing both.

5. Repeat Exercise 2, but add a solution `sol4` of the higher-order Taylor approximation to (2),

$$\frac{d^2\theta}{dt^2} + \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 = 0. \tag{3}$$

Comment on the result. Try also the approximation with the θ^3 term included but the θ^5 term omitted; why does this case give trouble?

6. Solve (1) numerically with $g/l = 1$ and $c = .5$, using the initial condition $\theta(0) = 0$, $\theta'(0) = 2$. Plot the solution in the θ, θ' plane. Repeat for different values of $\theta'(0)$ until you obtain the solution that corresponds to the pendulum converging to the vertically upright position.