Spring 1998

Maple Lab, Week 28

A Flask of Blood

Objective: This lab will improve your understanding of the ϵ–δ definition of a limit.

Review of Maple's equation-solving commands: Suppose you want to solve the equation $\frac{x}{\pi} + \sin x = 1$. You first enter the equation to make sure you have typed it properly:

```
>eq := x/Pi + sin(x) = 1;
```

You can then plot the left and right hand sides of the equation to see approximately where they intersect:

```
>plot({ lhs(eq), rhs(eq) }, x=-2*Pi..4*Pi);
```

You observe that there are three intersection points (solutions), one between 0 and 2, one between 2 and 4, and one between 4 and 6. To find the solutions more precisely, you might first try Maple's `solve` command:

```
>solve(eq, x);
```

Well, it seems (in Release 4) that Maple can't solve the equation exactly. So, use Maple's `fsolve` command to find approximate (numerical) solutions:

```
>fsolve(eq, x);
```

This gives only the solution between 2 and 4. (You can recognize it as π — an exact solution that `solve` did produce in Release 3!) After `fsolve` finds one solution, it stops. To find others, you need to add a range to the `fsolve` command to lead it to the solution you want:

```
>fsolve(eq, x=0..2);   fsolve(eq, x=4..6);
```

Now you have all three solutions.

Exercises:

1. The flask of blood

You are working in a medical lab where they measure the volume of blood in a conical flask by accurately measuring the height of the liquid. A particular flask is 5 cm in radius and 20 cm high. You can see a picture of it by executing the following Maple plot:

```
>plot3d([r*cos(theta), r*sin(theta), 4*r], r=0..5, theta=0..2*Pi,
    scaling=constrained, axes=normal, orientation=[30,75]);
```

The volume of a cone is given by

```
>V := 1/3 *Pi*r^2*h;
```

(Execute these commands as you read along.) For this cone, if the blood fills up to a height h, then the radius of the surface of the blood is

```
>r := h/4;
```

So the volume is:

```
>V;
```

Plot the volume as a function of the height.

Your medical supervisor says that the volume of the blood needs to be $V_0 = 40$ cm^3. She asks you to find the height h_0 of the blood in the flask. So execute

```
>V0 := 40;
```

and solve the equation $V = V_0$. Save the relevant root as `h0`.

Your supervisor now says that the volume of blood needs to be accurate to ± 1 cm^3. Let this "output accuracy" or "tolerance" be called ϵ:

```
>epsilon := 1;
```

Then she asks you how accurately the height of the blood needs to be measured. This "input accuracy" is called δ. You need to find δ so that if the height h is in the interval $(h_0 - \delta, h_0 + \delta)$ then the volume V will be in the interval $(V_0 - \epsilon, V_0 + \epsilon)$. Let's approach this problem graphically before getting into algebra. First execute

```
>Vp := V0+epsilon;    Vm := V0-epsilon;
```

Add the horizontal lines at V_p and V_m to your plot of V. Then click with the mouse to find the values h_p and h_m where $V = V_p$ and $V = V_m$ respectively. (You can improve your results by restricting the domain in the plot to `h=8..9` or even further.)

In your plot (with h restricted to $[8, 9]$) you can see that V is contained in the interval (V_m, V_p) exactly when h is contained in the interval (h_m, h_p). (**Note:** This would not necessarily be so if the function $V(h)$ were not a monotonic (either increasing or decreasing) function.) Now recall that we are seeking the largest value of δ such that $V(h)$ is in (V_m, V_p) when h is contained in $(h_0 - \delta, h_0 + \delta)$. Therefore, we are trying to find the largest value of δ such that the interval $(h_0 - \delta, h_0 + \delta)$ is contained in the interval (h_m, h_p). Since h_p and h_m do not have to be equidistant from h_0, δ must be the **smaller** of $\delta_p = h_p - h_0$ and $\delta_m = h_0 - h_m$.

To get an accurate value of δ, use `fsolve` to solve the equations $V = V_p$ and $V = V_m$ (saving the results as `hp` and `hm`), then use Maple to compute δ_p, δ_m, and $\delta = \min(\delta_p, \delta_m)$. (Maple has a `min` function that can be used here.)

Your supervisor now decides that the blood must be accurate to within ± 0.05 cm^3. Repeat your calculations for this case. (**You don't need to retype everything**. Just change `epsilon` and reexecute the commands from that point on.)

The fact that you can compute a satisfactory δ for any ϵ is summarized in the statement that

$$\lim_{h \to h_0} V(h) = V_0 \,.$$

This limit means:

> For every $\epsilon > 0$ there is a $\delta > 0$ such that
> if the height h is in the interval $(h_0 - \delta, h_0 + \delta)$ (with $h \neq h_0$)
> then the volume V will be in the interval $(V_0 - \epsilon, V_0 + \epsilon)$.

2. The famous trig limit

Consider the function $f(x) = \dfrac{\sin x}{x}$.

```
>f := x -> sin(x)/x;
```

The function is undefined at $x = 0$. Maple agrees:

```
>f(0);
```

However, the function does have a limit as x approaches 0. To see this, plot the function

```
>plot(f);
```

and use it to guess whether $\lim_{x \to 0} \dfrac{\sin x}{x}$ exists and, if so, what its numerical value is. **Note:** As far as Maple and graphics are concerned, your guess is truly a guess. When Maple plotted the function, it simply plotted some points and connected the dots. (To see these dots, click in the plot window menu on *Style* and *Point*.) So there is no guarantee that between 0 and the closest dot the function won't become much larger or much smaller. Of course, you know that for this well-known function that doesn't happen; but a computer-generated graph is not a proof of that.

Check that Maple agrees with your limit by executing

```
>Limit(f(x), x=0);   value(");
```

This limit means:

> For every $\epsilon > 0$ there is a $\delta > 0$ such that
> if x is in the interval $(-\delta, \delta)$ (with $x \neq 0$)
> then $\frac{\sin x}{x}$ is guaranteed to be in the interval $(1 - \epsilon, 1 + \epsilon)$.

Given that $\epsilon = 0.2$, find the largest δ such that if $0 < |x| < \delta$ then $\left|\frac{\sin x}{x} - 1\right| < \epsilon$. Again you can do this roughly by adding horizontal lines to your plot and using the mouse to identify the points x_p and x_m where $f(x)$ leaves the interval $(1-\epsilon, 1+\epsilon)$. Then use `fsolve` to identify δ more accurately. (Comment on how this situation is different from the case of the flask.)