Math. 302 (Fulling) 17 December 2002

Final Examination – Solutions

Name: ______________________________ Number: ______________
(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* Let a_n be the number of ways to distribute n apples to 3 children so that no child gets fewer than 2 apples and no child gets more than 5 apples. Find the generating function for a_n and use it to compute a_{10}.

The generating function for the possible results for one child is

$$x^2 + x^3 + x^4 + x^5 .$$

The generating function for the problem is the cube of that,

$$\begin{aligned}(x^2 + x^3 + x^4 + x^5)^2 &= x^6(1 + x + x^2 + x^3)^3 \\ &= x^6 \left(\frac{1 - x^4}{1 - x}\right)^3 .\end{aligned}$$

We need the coefficient of x^{10} in its Maclaurin expansion, hence the coefficient of x^4 in

$$\begin{aligned}\left(\frac{1 - x^4}{1 - x}\right)^3 &= (1 - x^4)^3 \sum_{j=0}^{\infty} \binom{3 - 1 + j}{j} x^j \\ &= [1 - 3x^4 + O(x^8)] \left[1 + 3x + \cdots + \frac{3 \cdot 4 \cdot 5 \cdot 6}{4!} x^4 + \cdots\right] \\ &= \cdots + (-3 + 15)x^4 + \cdots .\end{aligned}$$

Thus $a_{10} = 12$. (Here $O(x^8)$ stands for the irrelevant terms of degree 8 or higher; O refers here to the asymptotic behavior near 0, not ∞.)

2. *(40 pts.)* Solve these recursion relations.

(a) $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 5$, $a_1 = 1$.

Try $a_n = r^n$, getting the condition

$$0 = r^2 - r - 2 = (r + 1)(r - 2).$$

Thus the general solution is $a_n = b_1(-1)^n + b_2 2^n$. From the initial data, we must have

$$\begin{aligned} 5 &= b_1 + b_2 , & b_1 &= 3 , \\ 1 &= -b_1 + 2b_2 ; & b_2 &= 2 . \end{aligned}$$

Thus

$$a_n = 3(-1)^n + 2 \cdot 2^n = 3(-1)^n + 2^{n+1} .$$

(b) $a_n = 6a_{n-1}$, $a_0 = 7$.

$$a_n = 7 \cdot 6^n .$$

(c) $a_n = 6a_{n-1} - 9a_{n-2}$. [Find the *general solution*.]

In the usual way, we get

$$0 = r^2 - 6r + 9 = (r-3)^2 .$$

Since there is only one root, we must multiply the solution r^n by n to get a second independent solution. The general solution is

$$a_n = b_1 3^n + b_2 n 3^n .$$

(d) $a_n - a_{n-1} = n$, $a_0 = 0$.

Method 1: Temporarily write n as j : $a_j - a_{j-1} = j$. Sum this equation from $j = 1$ to $j = n$:

$$a_n - a_0 = \sum_{j=1}^{n} j = \frac{n(n+1)}{2} .$$

Thus $a_n = \frac{1}{2} n(n+1)$.

Method 2: First solve the corresponding homogeneous equation, $a_n^{\mathrm{h}} - a_{n-1}^{\mathrm{h}} = 0$. It's obvious that the solution is independent of n , and indeed the official maneuver of trying $a_n^{\mathrm{h}} = r^n$ yields $r - 1 = 0$, hence $a_n^{\mathrm{h}} = r^1 = 1$ (times an arbitrary constant). If we now seek a particular solution of the nonhomogeneous equation, we must start with $An + B$ (because the nonhomogeneous term contains n) and then promote it to

$$a_n = An^2 + Bn$$

(because the homogeneous solution is 1). Substituting into the recursion relation, we find

$$\begin{aligned} n &= An^2 + Bn - A(n-1)^2 - B(n-1) \\ &= A(2n-1) + B \\ &= 2An + (B - A) . \end{aligned}$$

Thus

$$\begin{aligned} 2A &= 1 , & A &= \tfrac{1}{2} , \\ B - A &= 0 ; & B &= \tfrac{1}{2} . \end{aligned}$$

Thus $a_n = \frac{1}{2} n^2 + \frac{1}{2} n + C$. The initial condition $0 = a_0 = C$ gives us (the hard way) the same solution as before.

3. *(24 pts.)* Let $\mathcal{O}(n)$ stand for "n is an odd integer". Consider the statement

p : If the product of two integers is odd, then both integers are odd.

(a) Rewrite p in symbols ($\mathcal{O}$, quantifiers, and logical connectives).

$$\forall m \, \forall n \, [\mathcal{O}(mn) \ \rightarrow \ \mathcal{O}(m) \wedge \mathcal{O}(n)] .$$

(b) State the *converse* of p both in symbols and in smooth English.

$$\forall m\,\forall n\,[\mathcal{O}(m)\wedge\mathcal{O}(n) \;\rightarrow\; \mathcal{O}(mn)]\,.$$

The product of two odd integers is always odd.

(Of course, considerable variation in the English sentences is possible.)

(c) State the *negation* of p both in symbols and in smooth English.

$$\exists m\,\exists n\,[\mathcal{O}(mn)\wedge\neg\big(\mathcal{O}(m)\wedge\mathcal{O}(n)\big)]\,.$$

(Rewriting the last clause as $\big(\neg O(m)\wedge\neg O(n)\big)$ is optional.)

There exist two integers, not both odd, whose product is odd.

(d) Which of those three statements is/are true?

The proposition and its converse are true. ("Odd" means "not divisible by 2".) The negation is false.

4. *(14 pts.)* Show (by any method) that $\displaystyle\sum_{k=0}^{700}\binom{700}{k} = 2^{700}$.

You've seen this one before. See the solutions for Test A, Question 1(b).

5. *(28 pts.)* For each of these "divide and conquer" recursions, either find an asymptotic estimate on $T(n)$ using the master theorem, or explain why the master theorem does not apply. *Hint:* What is 4^3?

(a) $\displaystyle T(n) = 3T\left(\frac{n}{2}\right) + n\,.$

Because $L\equiv\log_2 3 > 1$, one has $s(n)\equiv n\in O(n^{L-\epsilon})$, so $T\in\Theta(n^L)$ by Part 1 of the theorem.

(b) $\displaystyle T(n) = 8T\left(\frac{n}{4}\right) + \frac{n\sqrt{n}}{\lg n}\,.$

$L\equiv\log_4 8 = \frac{3}{2}$, since $4^{3/2} = 2^3 = 8$. Now we observe that $s(n)\equiv n^L/\lg n\notin\Theta(n^L)$, but $s(n)\notin O(n^{L-\epsilon})$ either. Therefore, the theorem doesn't apply.

(c) $\displaystyle T(n) = 8T\left(\frac{n}{4}\right) + \frac{n\sqrt{n}}{2}\,.$

Again $L=\frac{3}{2}$. This time $s(n)\equiv\frac{1}{2}n^L\in\Theta(n^L)$, so $T\in\Theta(n^{3/2}\lg n)$ by Part 2 of the theorem.

6. *(20 pts.)* [Leave answers in terms of factorials.]

(a) How many permutations of the English alphabet (26 letters) contain the block YES?

Treat YES as a single letter. There are 23 other letters. The number of permutations is $24!$.

(b) What is the probability that a randomly chosen permutation of the alphabet contains neither YES nor NO?

24! permutations contain YES, and by the analogous argument 25! contain NO. But some of these contain *both*, and we don't want to count those twice. If YES and NO are treated as single letters, there are $26-5+2$ letters and hence 23! such permutations. Therefore, the probability of avoiding YES and NO is

$$\frac{26!-25!-24!+23!}{26!}\,.$$

7. *(15 pts.)* Define a sequence by

$$t_0 = 1\,, \quad t_1 = 2\,, \quad t_2 = 4\,, \qquad t_n = t_{n-1} + t_{n-2} + t_{n-3} \quad \text{for } n \geq 3\;.$$

Prove by induction that $t_n \geq 1.5^n$ for all $n \geq 0$.

Base: $1.5^0 = 1 = t_0\,,\qquad 1.5^1 = \frac{3}{2} < 2 = t_1\,,\qquad 1.5^2 = \frac{9}{4} < 4 = t_2\,.$

Induction: For $n > 2$,

$$\begin{aligned} t_n &= t_{n-1} + t_{n-2} + t_{n-3} \\ &\geq 1.5^{n-1} + 1.5^{n-2} + 1.5^{n-3} \\ &= 1.5^{n-3}\left(1 + \frac{3}{2} + \frac{9}{4}\right) \\ &= 1.5^{n-3}\cdot\frac{19}{4} \\ &\geq 1.5^{n-3}\cdot\frac{27}{8} = 1.5^n. \end{aligned}$$

8. *(20 pts.)* Let F be this set of functions:

$$\left\{ n + \ln n\,,\ 2^n\,,\ \sqrt{n^3+1}\,,\ \frac{n^2+1}{n+2}\,,\ 4^n\,,\ \frac{n^2+1}{\sqrt{n}} \right\}$$

Let $\mathcal{R}$ be the equivalence relation defined on F by

$$(f,g) \in \mathcal{R} \iff f \in \Theta(g)\,.$$

(a) What is the partition of F induced by $\mathcal{R}$? [Answer should be a *list of subsets ("cells") of* F.]

Note that $2^n = e^{n\ln 2}$ and $4^n = e^{n\ln 4}$ are in separate cells. The other two cells are easily identified as $\Theta(n)$ and $\Theta(n^{3/2})$. So the partition is

$$\left\{ n + \ln n, \frac{n^2+1}{n+2} \right\}, \qquad \{2^n\}, \qquad \{4^n\}, \qquad \left\{ \sqrt{n^3+1}, \frac{n^2+1}{\sqrt{n}} \right\}.$$

(b) Take one representative from each cell of this partition and order these representatives from the slowest to the fastest (in terms of growth at infinity).

$$n + \ln n \prec \sqrt{n^3+1} \prec 2^n \prec 4^n$$

9. *(24 pts.)* Let A be a set and let $\mathcal{U}$ be the set of all subsets of A. ($\mathcal{U}$ is sometimes denoted by $\mathcal{P}(A)$ or 2^A.) Define a relation $\mathcal{R}$ on $\mathcal{U}$ by

$$(X,Y) \in \mathcal{R} \iff X \cap Y = \emptyset\,.$$

Answer the follow questions, **with justifications**:

(a) Is $\mathcal{R}$ reflexive?

NO — $X \cap X = \emptyset$ is never true except for $X = \emptyset$.

(b) Is $\mathcal{R}$ antisymmetric?

NO — $X \cap Y = \emptyset$ requires, rather than forbids, $Y \cap X = \emptyset$.

(c) Is $\mathcal{R}$ symmetric?

YES — see justification for (b).

(d) Is $\mathcal{R}$ transitive?

NO — if $X \neq \emptyset$ but $X \cap Y = \emptyset$, then $Y \cap X = \emptyset$, but $X \cap X = \emptyset$ does not follow.

(e) Is $\mathcal{R}$ an equivalence relation?

NO — it is not reflexive nor transitive.

(f) Is $\mathcal{R}$ an ordering?

NO — it is not antisymmetric nor transitive, so it isn't a strict ordering (not to mention an inclusive one, which also requires reflexivity).

Remark: Above we have tacitly assumed that $A \neq \emptyset$. In the trivial special case where A itself is the empty set, the answer to all the questions is YES!