Logic

In mathematical work we often need a better language or notation than ordinary English for expressing relationships among various assertions or hypothetical states of affairs. That is what is provided by modern *symbolic logic*. Like algebra, which provides a better way than ordinary English to talk about numbers, and therefore also provides *mechanisms for doing calculations*, symbolic logic is not just a language, but *a machinery that makes it easier to construct arguments or proofs and easier to test their validity*.

In a calculus course you may have seen some definitions or theorems like this:

> The function f is continuous if, for every x in the domain of f, for every number $\epsilon > 0$ there is a number $\delta > 0$ such that for any number y in the domain, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. (1)

Such "ϵ–δ" statements have the reputation of being extremely difficult for students. One reason is that the sentences seem very tangled and complex. We are asking ordinary language to do a job that it was not designed to do, much like writing out

$$x^2 - 3x + 2 = 0$$

as

> If a number is multiplied by itself, and then three times the number is subtracted and two is added, and the result is zero, what was the original number?

A solid grounding in formal logic would make it much easier for freshmen to understand the *limit* concept that is the foundation of calculus.

So, why isn't calculus taught that way? Probably the best reason is that logic is more abstract than the calculus itself, so students would not understand or appreciate it if college mathematics led off with it. You can't understand a solution until you first understand why there is a problem. To appreciate abstraction you first need some painful experience with the more concrete problems that it clarifies.

One of our present tasks is to discuss the logical structure of complex sentences such as (1) and introduce a mathematical notation for such structure. The irony here is that we are trying to overcome a difficult abstraction with an even deeper dose of abstraction. Fortunately, this abstraction does not require any quantitative intuition to understand, and it can be illustrated (somewhat artificially) by simple sentences from everyday life as well as by the mathematical statements that are the real focus of our interest.

Quantifiers

We present symbolic logic in the notation common among mathematicians as "blackboard shorthand". The notation used by professional logicians (who are usually located in philosophy departments) is sometimes slightly different.

$\forall x$ means "For all x":

$$\forall x\,[x < x + x^2 + 1] \qquad \text{(real numbers understood)} \tag{2}$$

is a true statement. $\exists x$ means "There exists an x such that," as in[1]

$$\exists x\,[x \text{ is rotten in Denmark}]. \tag{3}$$

$\forall x$ and $\exists x$ are called *universal* and *existential quantifiers*, respectively.

Expressions such as

$$x < x + x^2 + 1 \qquad \text{and} \qquad x \text{ is rotten in Denmark}$$

(which contain a variable and would be sentences if the variable were replaced by a meaningful name or noun) are called *open sentences*. They are just like *functions* or *formulas* in algebra and calculus, except that the *value* of such a function, when something particular is plugged in for x, is not a number, but rather a *truth value* — either True or False (or either "Yes" or "No").

A quantifier, $\forall x$ or $\exists x$, closes off an open sentence and turns it into a genuine sentence, which is either true or false (although we may not know which). Quantifiers are very much like the definite integral and limit notations in calculus, which turn formulas into numbers:

$$\int_0^1 x^2\,dx \qquad \text{and} \qquad \lim_{x\to 2} x^2$$

are particular numbers, even though the expressions representing them involve a variable, x. In calculus such a variable is often called a "dummy variable"; in logic it's traditionally called a "bound variable" (because it's tied to its quantifier).

The quantified variable stands for objects in some "universe of discourse", which may be either stated explicitly —

$$\forall x\,[\text{if } x \text{ is a real number, then } x < x + x^2 + 1] \tag{2'}$$

[1] Of course, the usual expression is, "Something is rotten in Denmark," not "There is an x such that x is rotten in Denmark." This brings out the point that there are many ways of saying the same thing in English, often with slightly different connotations. No one way is the only *right* way. In addition to "for all," universal quantifiers can be translated "for each," "for every," or "for any." Similarly, for an existential quantifier we may say "There is an x for which . . . ," or just "Some x . . . [does so-and-so]." Furthermore, to make a natural English sentence, we sometimes use common sense to rephrase a logical construction rather drastically. (For example, compare sentences (6′) and (6) below.)

— or understood from context. The universe of discourse is just like the "domain" of a numerical function. (Don't think of "universe" in the astronomical sense.)

Leading universal quantifiers are often omitted when we state "identities" in mathematics, such as

$$x + y = y + x \qquad \text{and} \qquad (n+1)! = (n+1)n!\,. \tag{4}$$

[What universe of discourse is understood from context in each of these cases?]

If two or more quantifiers of the same type are adjacent, their order doesn't matter:

$$\forall x \forall y\,[x + y < y + x + 1] \qquad \text{and} \qquad \forall y \forall x\,[x + y < y + x + 1] \tag{5}$$

say exactly the same thing.

However, the order of quantifiers of different type is extremely important. (This is the place where this discussion gets beyond the obvious into something both important and subtle.) Consider, for example, the old saying

$$\text{Behind every successful man there is a woman.} \tag{6}$$

The structure of this proposition (whether or not you believe it to be true or false) is

$$\forall x \exists y\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x]. \tag{6$'$}$$

But

$$\exists y \forall x\,[\text{if } x \text{ is a man and } x \text{ is successful, then } y \text{ is a woman and } y \text{ is behind } x] \tag{7}$$

says something completely different: There is *one particular woman* who stands behind every successful man in the world! (There may be more than one, but each one of them deals with all the men.)

It is notorious that this point is important for the ϵ and δ in the definition of a limit, (1), which has this structure:

$$\forall x \forall \epsilon \exists \delta \forall y\,[\text{if } |y - x| < \delta, \text{ then } |f(y) - f(x)| < \epsilon] \tag{1$'$}$$

(with qualifications such as "number" and "in the domain" suppressed). For example, look at this slight variation of (1):

> For every x in the domain of f, there is a number $\delta > 0$ such that for every number $\epsilon > 0$ and every number y in the domain, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. (8)

Its structure is

$$\forall x \exists \delta \forall \epsilon \forall y\,[\text{if } |y - x| < \delta, \text{ then } |f(y) - f(x)| < \epsilon], \tag{8$'$}$$

and it is false unless f is a constant function![2] For a nontrivial f, one has to know ϵ before one can choose the right δ. There is not (usually) one δ that works for every ϵ. Similarly, if we move the universal quantifier $\forall x$ in (1) after the existential quantifier $\exists\delta$, we get a different condition:

$$\begin{aligned}&\text{For every number } \epsilon > 0 \text{ there is a number } \delta > 0 \text{ such that for every}\\&x \text{ and } y \text{ in the domain of } f\text{, if } |y-x| < \delta\text{, then } |f(y) - f(x)| < \epsilon.\end{aligned} \tag{9}$$

$$\forall\epsilon\exists\delta\forall x\forall y\,[\text{if } |y-x| < \delta,\text{ then } |f(y)-f(x)| < \epsilon]. \tag{9$'$}$$

Although this condition is simpler to state in English than the one in the definition (1), it is *harder* for the function f to satisfy, because it requires that the *same* δ work for all x. For example, the function $f(x) = 1/x$ is continuous on the domain $(0, \infty)$, but it does not satisfy (9) because its graph becomes increasingly steep as x approaches 0.[3]

If two quantifiers of one type are separated by one (or more) of the other type, then they cannot be reversed:

$$\exists x\forall y\exists z \quad \text{is not equivalent to} \quad \exists z\forall y\exists x; \tag{10}$$

"There is a country where behind every man there is a woman" is not equivalent to "There is a woman such that for every man there a country where she stands behind him."

Propositional calculus (the logical connectives)

Letters $p, q, \ldots$ are used as variables standing for statements[4] or for open sentences (sentences containing variables). In the latter case, the variable is sometimes made explicit by writing something like $p(x)$. (For example, we can let p, or $p(n)$, stand for "n is a perfect square" — i.e.,

$$\exists m\,[n = m^2]$$

— and q for "n is an even number." To discuss p in the context of quantifiers, we would write $\exists n\, p(n)$, etc., or even $\exists n\exists m\, r(n,m)$ to display the entire structure.) The second major part of logical notation expresses how simple sentences are combined into compound ones. The easiest of these to understand simply translate the English words "and", "or", and "not".

[2] Technicality: (8) might be true for a nonconstant function whose domain consists of discrete points, such as the integers, or even a function whose domain consists of separated intervals. For this example we have in mind the typical situation in calculus, where the domain is a single interval of real numbers.

[3] A function that *does* satisfy (9) is called *uniformly continuous*. Uniform continuity is an important condition in more advanced mathematics, but we are interested in it today only as an example of the need to keep track of the order of quantifiers. (By the way, Augustin-Louis Cauchy, who did as much as anyone to invent the concept of a limit, seems to have been confused on this point for at least 26 years, so students shouldn't expect to find it easy on the first exposure.)

[4] "Proposition" is just another word for "statement".

AND: $p \wedge q$ (n is a square and also is even.)

OR: $p \vee q$ (Either n is a square, or it is even (possibly both).)

NOT: $\neg p$ (n is not a perfect square.)

Note that $\neg\neg p$ simplifies to p. Also, it is easy to see that $\wedge$ and $\vee$ are commutative and associative operations, so we can write things like $p \wedge q \wedge r$ instead of $p \wedge (q \wedge r)$ and $(p \wedge r) \wedge q$. Furthermore, each of them is distributive over the other:

$$p \wedge (q \vee r) \quad \text{is equivalent to} \quad (p \wedge q) \vee (p \wedge r), \tag{11a}$$

$$p \vee (q \wedge r) \quad \text{is equivalent to} \quad (p \vee q) \wedge (p \vee r). \tag{11b}$$

(We shall prove one of these with truth tables in a moment.)

Truth tables

It is convenient and standard to let 1 represent "True" or "Yes" and 0 represent "False" or "No".

Each connective can be precisely defined by telling what its truth value is for each possible truth value of its parts. This information can be presented in *truth tables* similar to addition and multiplication tables in arithmetic:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

From these the truth tables of more complicated sentences can be deduced. For example, let's establish the distributive law (11a). We list all 8 possible cases:

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

From the fifth and eighth columns of this table, we see that the two sides of (11a) are true in exactly the same cases (the last 3); therefore, replacing one side by the other is a universally valid principle of logical reasoning.

De Morgan's laws

Two other logical identities that can be verified by truth tables are *De Morgan's laws*,

$$\neg(p \wedge q) \quad \text{is equivalent to} \quad \neg p \vee \neg q, \tag{12a}$$

$$\neg(p \vee q) \quad \text{is equivalent to} \quad \neg p \wedge \neg q. \tag{12b}$$

Let us leave the proofs of (11a), (12a), and (12b) to exercises. Instead, for variety, let's look now at what happens when we try to prove an identity that turns out to be incorrect. Suppose we conjectured that

$$\neg(p \wedge q) \quad \text{is equivalent to} \quad \neg p \wedge \neg q.$$

We work out the corresponding truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	0	1	1	0	0
1	0	0	1	0	1	0
1	1	1	0	0	0	0

The fourth and seventh columns do not match up, so the conjecture was wrong.

Closely related to De Morgan's laws are these laws for quantifiers:

$$\neg\forall x\,[\ldots] \quad \text{is equivalent to} \quad \exists x \neg[\ldots], \tag{13a}$$

$$\neg\exists x\,[\ldots] \quad \text{is equivalent to} \quad \forall x \neg[\ldots]. \tag{13b}$$

Note that for a *finite* universe, quantifiers are unnecessary — they can be rewritten in terms of connectives! If the universe has only 3 elements, say a, b, c, then

$$\forall x\, p(x) \quad \text{is equivalent to} \quad p(a) \wedge p(b) \wedge p(c),$$

$$\exists x\, p(x) \quad \text{is equivalent to} \quad p(a) \vee p(b) \vee p(c).$$

Then the action (13) of $\neg$ on quantifiers follows from DeMorgan's laws (12).

Both of these sets of laws are especially nice when there are negations on both sides:

$$\neg(\neg p \wedge \neg q) \quad \text{simplifies to} \quad p \vee q,$$

$$\neg\forall x\, \neg p(x) \quad \text{simplifies to} \quad \exists x\, p(x),$$

etc.

Connectives expressing logical equivalence and implication

We consider two more extremely important logical connectives:

p	q	$p \longleftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Observe:

1. $p \longleftrightarrow q$ says that p and q have the *same truth value* (either both true or both false). Therefore, if we know one, we can conclude the other. The distributive law (11a) can be expressed totally in symbolic notation as

$$p \wedge (q \vee r) \longleftrightarrow (p \wedge q) \vee (p \wedge r). \tag{11$'$}$$

2. $p \longleftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.

3. $p \rightarrow q$ is intended to symbolize that *if we know that p is true, then we can conclude that q is true.* Note that the bottom half of its truth table guarantees this. The most common English rendering of $p \rightarrow q$ is, "If p, then q." One also says "p implies q."

4. $p \rightarrow q$ is equivalent to $\neg p \vee q$. (By the way, like the minus sign in algebra, $\neg$ binds tightly to the propositional symbol it applies to ("has highest precedence"); for example, $\neg p \vee q$ means $(\neg p) \vee q$, not $\neg(p \vee q)$.)

With the full set of logical notation, we can now write the structure of the definition of continuity entirely in symbols:

$$[f \text{ is continuous}] \longleftrightarrow \forall x \forall \epsilon \exists \delta \forall y \, [p(x,y,\delta) \rightarrow q(x,y,\epsilon)]. \tag{1$''$}$$

The connective $\rightarrow$ is slightly subtle conceptually. Right now you may be wanting to ask:

1. What is the justification for the top half of the table? Does it make sense to say, "If $2+2=5$, then $\sin x$ is a continuous function," or "If $2+2=5$, then $\sin x$ is not a continuous function."?

2. Does it make sense to say, "If China is in Asia, then $\sin x$ is a continuous function," when the two statements obviously have no connection with each other?

3. In Grimaldi's book, what is the difference between the single-shafted arrows ($\rightarrow$, $\longleftrightarrow$) and the double-shafted arrows ($\Rightarrow$, $\Longleftrightarrow$)?

All these questions are related.

The answer to question 1 is that we want $p \;\rightarrow\; q$ to be meaningful and useful when p and q are *open sentences* — in particular, when they are inside *quantified* sentences, such as

$$\forall x \left[\text{If } |x| < \frac{\pi}{4}, \text{ then } |\sin x| < \frac{1}{\sqrt{2}}\,. \right] \tag{14}$$

This is a true and useful theorem. It does what we want of a theorem: Whenever a number is less than $\frac{\pi}{4}$ in magnitude, it enables us to conclude (correctly) that its sine is less than $\frac{1}{\sqrt{2}}$. However, there are other numbers, such as $x = \frac{\pi}{2}$, for which both the hypothesis and the conclusion are false, and there are still other numbers, such as $x = \pi$, for which the hypothesis is false but the conclusion is true. We must demand that these cases be *consistent with the theorem*, and the connective $\rightarrow$ is defined to make this so. If we changed the top two lines of the truth table, or left them undefined, then the theorem would become false, or indeterminate, for some of these cases. This would make the formulation of mathematical statements very cumbersome.

The resolution of point 2 is similar. Propositional calculus is concerned only with the *truth values* of sentences, not with what they *mean*. There are only 2 truth values, Yes and No. In this sense all true sentences are the same, and all false ones are the same, just as all numbers 9 are the same, regardless of what things you counted to get the number 9. Therefore, to say

If China is in Asia, then $\sin x$ is a continuous function. (15)

is no more strange than to say

The number of planets in the solar system is less than or equal
to the number of states in the Union. (16)

There is no scientific law that makes the latter statement true; it is simply a fact.

Some philosophical fine points: Object language vs. metalanguage

Now, about those arrows. (If they didn't bother you, feel free to ignore this subsection.) In algebra, we make *statements about numbers*. In logic, we make *statements about statements*, and this creates some new conceptual complications. In the technical language of logic, the connectives $\rightarrow$ and $\longleftrightarrow$ belong to the *object language*; they are always *part of a statement under discussion*. The symbols $\Rightarrow$ and $\Longleftrightarrow$ belong to the *metalanguage*; they are used *to talk about statements* and their relationships to each other. For instance,

$$p \wedge q \;\rightarrow\; q$$

is a statement (or statement framework) that we can write down, contemplate, test with a truth table, etc., whereas

$$p \wedge q \;\Rightarrow\; q$$

is an *operational principle*: it says that if I am given $p \wedge q$ as one of the premises in an argument, then I can validly conclude q. There is a similar distinction between $\longleftrightarrow$ and $\Longleftrightarrow$: see Grimaldi, Definition 2.2 (p. 56) as contrasted with paragraph 2d (p. 48).

Are these real distinctions? That is, is it ever possible for $s_1 \ \rightarrow\ s_2$ to be true while $s_1 \ \Rightarrow\ s_2$ is false, or vice versa?[5] Whenever we are discussing propositional calculus in the abstract, the only statements we can have in the roles of s_1 and s_2 are *propositional formulas* built up out of letters $p, q, \ldots$ and the logical connectives, without any explicit words or symbols relating to any particular subject matter. In this context, the only way we could possibly be justified in concluding s_2 from s_1 (hence asserting that $s_1 \ \Rightarrow\ s_2$) is to observe that $s_1 \ \rightarrow\ s_2$ is a *tautology* — that is, true regardless of the truth values assigned to the letters $p, \ldots$ that appear in it.[6] However, in discussions of scientific theories with axioms or background information that is not explicitly stated in the premises, the meaning of $\Longleftrightarrow$ depends on how much is tacitly assumed. For example,

$$\forall x[(\text{distance of } x \text{ from center of sun}) < 1000\,\text{km} \ \rightarrow\ (\text{temperature at } x) > 10,000\,^\circ\text{C}]$$

is true, because of facts of physics and astronomy, but not a tautology; correspondingly,

$$(\text{distance of } x \text{ from center of sun}) < 1000\,\text{km} \ \Rightarrow\ (\text{temperature at } x) > 10,000\,^\circ\text{C}$$

is true in the operational sense within the theoretical framework of generally accepted science, but is not true as a matter of pure logic.

A related distinction is that between 1 and 0, on the one hand, and T_0 and F_0, on the other, in Grimaldi's notation. According to p. 53, T_0 stands for any statement that is a tautology, whereas 1 stands for any statement that is true. In terms of propositional functions (see most recent footnote), T_0 is the constant function whose value is always 1, and F_0 is the function whose value is always 0.

Rest assured that in our class work, we will usually be able to use $\rightarrow$, $\longleftrightarrow$, 0, and 1 interchangeably with $\Rightarrow$, $\Longleftrightarrow$, T_0, and F_0 , respectively, without going seriously wrong.

Some terminology and language quirks

1. It is sometimes said that the English counterpart of $\rightarrow$ is IF, but this is not true in the same sense that the English counterpart of $\wedge$ is AND. Notice that $p \ \rightarrow\ q$ can be

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[5] Grimaldi's book is somewhat ambiguous on such issues, which is why I find it necessary to make such a fuss over them. Definition 2.2 suggests the *operational* interpretation of the double-shafted arrows, but there are remarks on pp. 48, 58, and 70–71 indicating that a double-shafted assertion is true if and only if the corresponding single-shafted statement is a tautology. The sentence below Definition 2.2 shows that at that point the only statements the author is considering are propositional formulas; therefore, the two interpretations are equivalent in that temporary context. But as soon as quantifiers are introduced (Sec. 2.4), the issue needs to be reopened; at the very least, "tautology" needs to be redefined.

[6] Another way of saying this is that a propositional formula in, say, two variables p and q represents a function $f(p, q)$ that takes on only the values 0 and 1 as p and q themselves range over those values. As pointed out in Example 15.26 (pp. 688–689 of Grimaldi), there are exactly 16 such functions. [Why 16?] The two formulas $p \vee q$ and $q \vee p$ represent the same propositional function; this assertion is equivalent to $p \vee q \ \Longleftrightarrow\ q \vee p$ and also equivalent to the assertion that $p \vee q \ \longleftrightarrow\ q \vee p$ is a tautology.

expressed as

$$\text{If } p, \text{ then } q, \tag{17}$$

but that

$$p \text{ IF } q$$

corresponds instead to $q \Rightarrow p$. However, if we say

$$p \text{ ONLY IF } q, \tag{17$'$}$$

then we do get something that means $p \rightarrow q$; it says that if q is false, then p is false, which is the contrapositive (see 3 below) of $p \rightarrow q$. A consequence of this is that

$$p \text{ IF AND ONLY IF } q \tag{18}$$

is an English way of saying $p \longleftrightarrow q$. In fact, it is the standard way of expressing equivalent conditions (usually $\forall x\,[p(x) \longleftrightarrow q(x)]$ statements) in mathematical English. Often it is "blackboard abbreviated" to "IFF".

2. If $p \Rightarrow q$, or $\forall x\,[p(x) \rightarrow q(x)]$, then one says

$$p \text{ is a \textit{sufficient condition} for } q \tag{17$''$}$$

and

$$q \text{ is a \textit{necessary condition} for } p \tag{17$'''$}$$

(that is, if q is false, then p can't be true). Therefore, if p is both necessary and sufficient for q, then $p \iff q$ or $\forall x\,[p(x) \longleftrightarrow q(x)]$.

3. Associated with $p \rightarrow q$ are 3 closely related propositions:

$$\begin{array}{cc} \text{contrapositive:} & \neg q \Rightarrow \neg p \\ \text{converse:} & q \Rightarrow p \\ \text{inverse:} & \neg p \Rightarrow \neg q \end{array}$$

The original statement and its contrapositive are logically equivalent. The converse and the inverse are equivalent, because the inverse is the contrapositive of the converse. But the original and the converse are *not* logically equivalent (although they may both be true under certain circumstances).

Example of 2 and 3: Another bugaboo of the calculus student is *infinite series*, $\sum_{n=0}^{\infty} a_n$. Let p stand for

$$a_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{19}$$

and q stand for

$$\text{The series } \sum_{n=0}^{\infty} a_n \text{ converges.} \tag{20}$$

Then q implies p (Stewart, Sec. 10.2, Theorem 6), but p does *not* imply q (because the harmonic series, for instance, is a counterexample: Stewart Sec. 10.2, Example 7). Thus

(A) Tending of the terms to 0 is a *necessary* condition for convergence, but not a *sufficient* condition. (In contrast, most of the series convergence theorems state *sufficient* conditions (e.g., the alternating series test (Stewart, Sec. 10.4) and the ratio test (Sec. 10.3).)

(B) $p \to q$ is false for the harmonic series, but its converse, $q \to p$, is true. (Of course, $p \to q$ is true of all series that happen to be convergent.) If we attach a universal quantifier over all series, $a \equiv \{a_n\}$, then the result

$$\forall a\, [p(a) \;\to\; q(a)] \tag{21}$$

is false, but

$$\forall a\, [q(a) \;\to\; p(a)] \tag{22}$$

is true.

4. A standard logical terminology is

$\to$	conditional
$\longleftrightarrow$	biconditional
$\Rightarrow$	implication
$\Longleftrightarrow$	equivalence

However, Grimaldi uses "implication" for "conditional". Don't worry about that.

5. **Counterfactual conditionals.** Contemplate this sentence:

> If the Federal Reserve had not lowered the interest rate, we would be in a recession now. (23)

Is it *automatically true* by the truth table for $\to$, if the Fed *did* lower the interest rate? That doesn't seem right. Now consider this one:

> If Napoleon and Julius Caesar had been contemporaries, then Napoleon would be 2000 years old by now. (24)

Philosophers still argue over sentences like these. In my opinion, they are meaningful only when there is a (tacit) understanding of *what remains the same* in the hypothetical changed situation. In case (24), are we changing Napoleon's birthday or Caesar's, or both? A technically precise situation where such an issue arises is the definition of a *partial derivative*, where one must be clear on which other variables are held fixed:

$$\left(\frac{\partial f}{\partial r}\right)_\theta \text{ (in polar coordinates)} \quad \text{and} \quad \left(\frac{\partial E}{\partial T}\right)_V \text{ (in thermodynamics)}$$

are not the same things as

$$\left(\frac{\partial f}{\partial r}\right)_x \quad \text{and} \quad \left(\frac{\partial E}{\partial T}\right)_P .$$

Exercises

1. In each part of (4), what universe of discourse is understood from context? (For the first part, at least, more than one correct answer is possible.)

2. Express in logical notation (quantifiers and connectives):

   (a) Everybody loves a lover.

   (b) If something quacks like a duck and waddles like a duck, then it is a duck.

   (c) All that glitters is not gold. (This one is worth a paragraph of discussion!)

3. Express in logical notation (quantifiers and connectives):

   (a) n is an even number.

   (b) For every number that is a perfect cube, there is a larger number that is even.

4. Express in logical notation (quantifiers and connectives):

   (a) Every U.S. citizen, and any person who earns income in the United States, must file a tax return.

   (b) Taxpayer Smith may claim his daughter as a dependent, provided that she is not married and filing a joint return, if he provides more than half of her support, or if he and another person together provide more than half of her support and he paid over 10% of her support. (Let $s =$ Smith, $d =$ daughter.)

5. Translate into standard mathematical English:

$$\forall n \, \forall a \, \forall b \, \big[(n < a \wedge n < b) \ \rightarrow \ n < ab\big].$$

   (A really good answer uses no letters for variables and is brief.)

6. Translate into standard mathematical English:

$$\exists f \forall x \, \exists y \, \forall z \, \big[z < x \ \Rightarrow \ f(z) \leq f(x) + y\big].$$

   (This time variables are allowed.)

7. Use truth tables to establish the validity of the other distributive law (11b),

$$p \vee (q \wedge r) \quad \text{is equivalent to} \quad (p \vee q) \wedge (p \vee r).$$

   (Thus AND and OR are each distributive over the other, unlike addition and multiplication!)

8. Use truth tables to prove De Morgan's laws, (12a) and (12b).

9. Explain why
$$\neg\forall x\,[\neg p(x)] \quad \text{is equivalent to} \quad \exists x\, p(x).$$

10. Why are there exactly 16 propositional functions of two propositional variables, p and q?

11. We stated that (8) is true only for constant functions. Verify and strengthen this claim as follows: Show that (for a given x)
$$\exists\delta\forall\epsilon\forall y\,[|y-x|<\delta \;\Rightarrow\; |f(y)-f(x)|<\epsilon]$$
if and only if
$$\exists z\exists\epsilon\forall y\,[|y-x|<\epsilon \;\Rightarrow\; f(y)=z].$$
(The latter condition is expressed in mathematical English as "f is constant on some neighborhood of x.") The universe of discourse for ϵ and δ is the positive real numbers; the universe of discourse for x, y, and z is the domain of f.

12. Demonstrate by numerical examples that $f(x)=1/x$ satisfies (1) but not (9). (Given an ϵ and a δ, show that x and y can be put so close to 0 that $|y-x|<\delta$ but $|f(y)-f(x)|>\epsilon$.)