Combinatorics of multivariable polynomials

Monomials, the ingredients of polynomials, are algebraic expressions such as x^2y^3z, or, in generality, ${x_1}^{n_1}{x_2}^{n_2}\cdots{x_d}^{n_d}$. They give rise to three related combinatorial questions that are prototypes of problems that arise in many other contexts. One of these questions gives rise to the multinomial coefficients, another to the formula for the number of ways of distributing indistinguishable objects into boxes.

The simplest example

I) How many (homogeneous) quadratic monomials in 2 variables are there? (How many terms can a quadratic polynomial have?) We list them:

$$x^2, \quad y^2, \quad xy \quad : \quad 3 \text{ items.}$$

II) How many times does each monomial occur in $(x+y)^2$? Note that each *noncommutative* monomial occurs once:

$$x^2, \quad xy, \quad yx, \quad y^2 \quad : \quad 4 \text{ items.}$$

So we are asking how many noncommutative monomials (different orderings of the factors) reduce to the same commutative monomial. The answer is different for different exponent structures:

$$x^2,\ y^2 \ :\ 1 \text{ each.} \qquad xy\ :\ 2.$$

III) How many noncommutative monomials are there in all?

$$1+2+1=4 \quad \text{(sum of a row of Pascal's triangle)}$$

(In fact, we already saw this at the first step of answering Question II.)

Remark: The general polynomial of this type could be written

$$ax^2+bxy+cy^2,$$

or

$$Ax^2+B_1xy+B_2yx+Cy^2=Ax^2+2Bxy+Cy^2$$

where $B=\frac{1}{2}(B_1+B_2)$ and we might as well take $B_1=B_2=B$. If we define $x_1=x$, $x_2=y$, $A_{11}=A$, $A_{12}=A_{21}=B$, $A_{22}=C$, the polynomial becomes

$$\sum_{i=1}^{2}\sum_{j=1}^{2}A_{ij}x_ix_j\,.$$

(Linear algebra students know that these coefficients form a *matrix*,

$$(A_{ij})=\begin{pmatrix}A & B\\ B & C\end{pmatrix}.)$$

Knowing the answer to question II allows us to replace the redundant sum over ordered pairs of indices (i,j) by a sum over inequivalent terms:

$$A_{11}{x_1}^2+2A_{12}x_1x_2+A_{22}{x_2}^2.$$

Note that the index notation promises to ease the generalization to more variables or higher degrees or both.

Cubic (or higher) polynomials in 2 variables

I) How many monomials are there? Put 3 (or n) objects (the elementary linear factors) into 2 boxes (labeled by the variables). We calculate all arrangements of n objects and $2-1=1$ divider, the objects being indistinguishable: $\binom{n+2-1}{n}=n+1$, which is 4 when $n=3$. We can easily list them:

$$x^3, \quad x^2y, \quad xy^2, \quad y^3.$$

II) How many noncommutative monomials reduce to $x^{n_1}y^{n_2}$ with $n_1+n_2=n$? The answer is the binomial coefficient

$$\frac{n!}{n_1!\,n_2!}=\frac{n!}{n_1!\,(n-n_1)!}=\binom{n}{n_1}.$$

For $n=3$:

$$\begin{array}{lcccc} (n_1,n_2)=(3,0) & : & \frac{3!}{3!\,0!}=1 & : & x^3 \\ (n_1,n_2)=(2,1) & : & \frac{3!}{2!\,1!}=3 & : & x^2y,\ xyx,\ yx^2 \end{array}$$

and two more cases symmetrical to these.

III) How many noncommutative monomials in all? The easy way is to note that each factor chooses a variable, which is 2^n choices. The hard way is to sum up the answer to II:

$$\sum_{j=0}^{n}\binom{n}{j}=2^n \quad \text{(sum of a row of Pascal's triangle).}$$

For $n=3$, this is $8=1+3+3+1$. The answer to I is the number of terms in this sum.

Quadratic monomials in 3 (or more) variables

I) How many monomials? Put 2 objects (factors) into 3 (or d) boxes (variables): $\binom{2+d-1}{2}$, which is 6 when $d=3$. Check by listing them:

$$x^2, \quad y^2, \quad z^2, \quad xy, \quad xz, \quad yz$$

II) How many cases with the structure ${x_1}^{n_1}{x_2}^{n_2}\cdots{x_d}^{n_d}$ with $n_1+n_2+\cdots+n_d=2$? The answer is the multinomial coefficient $\frac{2!}{n_1!\,n_2!\cdots n_d!}$. For $d=3$ the only essentially different cases are

$$\begin{array}{lcc} (2,0,0) & : & 1 \\ (1,1,0) & : & 2 \end{array}$$

(Each of them occurs 3 times.)

III) How many in all? Each factor chooses a variable, so the answer is d^2. For $d=3$, we get $9=1+1+1+2+2+2$, so this checks against II. In general, we get a generalized "Pascal's pyramid" identity,

$$\sum_{\substack{n_1+n_2+\cdots+n_d=2 \\ n_i\geq 0}}\frac{2!}{n_1!\,n_2!\,\cdots\,n_d!}=d^2$$

The general case: nth-degree polynomials in d variables

I) How many monomials?

$$\binom{n+d-1}{n}.$$

II) How many cases with structure ${x_1}^{n_1}{x_2}^{n_2}\cdots{x_d}^{n_d}$ with $n_1+n_2+\cdots+n_d=n$?

$$\frac{n!}{n_1!\,n_2!\,\cdots\,n_d!}\,.$$

III) How many noncommutative monomials in all?

$$d^n = \sum_{n_i\ge 0}^{n_1+n_2+\cdots+n_d=n} \frac{n!}{n_1!\,n_2!\,\cdots\,n_d!}\,.$$

There is one term for each monomial in I. One way to get this equation is to evaluate $(x_1+\cdots+x_d)^n$ with all $x_i=1$.

The general polynomial $\sum_{i_1,i_2,\ldots i_n} A_{i_1 i_2 \ldots i_n} x_{i_1} x_{i_2}\cdots x_{i_n}$ can be written as a sum over all *inequivalent* index strings, weighted by the multinomial coefficients:

$$\sum_{\mathbf{n}}^{|\mathbf{n}|=n} \frac{n!}{\mathbf{n}!}\,A_{\mathbf{n}} x^{\mathbf{n}}$$

where

$$\mathbf{n}=(n_1,\ldots,n_d), \quad |\mathbf{n}| = \sum_{i=1}^{d} n_i\,, \quad \mathbf{n}! = \prod_{i=1}^{d} n_i!, \quad x^{\mathbf{n}} = {x_1}^{n_1}\cdots{x_d}^{n_d}$$

Derivatives

How many 2nd-order partial derivatives in 2 variables are there?

$$\frac{\partial^2 f}{\partial x^2}\,, \quad \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$$

It is easy to see that derivative operators will behave just like monomials; all the same questions can be asked and answered in the same way. (Fourier and Laplace transforms change derivatives into multiplications by new variables, so this *had* to work out the same way for consistency.)