

Matrices and Graphs of Relations

[THE GIST OF SEC. 7.2 OF GRIMALDI]

If $|A| = n$ and $|B| = p$, and the elements are *ordered and labeled* ($A = \{a_1, a_2, \dots, a_n\}$, etc.), then any relation \mathcal{R} from A to B (i.e., a subset of $A \times B$) can be represented by a matrix with n rows and p columns: M_{jk} , the element in row j and column k , equals 1 if $a_j \mathcal{R} b_k$ and 0 otherwise. Such a matrix is somewhat less inscrutable than a long list of ordered pairs.

Example: Let $A = B = \{1, 2, 3, 4\}$. The relation

$$\mathcal{R} = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 1), (3, 4)\}$$

has the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

BOOLEAN ARITHMETIC

Recall the operation tables for logical AND and OR:

| | | |
|----------|---|---|
| \wedge | 0 | 1 |
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| | | |
|--------|---|---|
| \vee | 0 | 1 |
| 0 | 0 | 1 |
| 1 | 1 | 1 |

Let's compare this with arithmetical TIMES and PLUS:

| | | |
|----------|---|---|
| \times | 0 | 1 |
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| | | |
|-----|---|---|
| $+$ | 0 | 1 |
| 0 | 0 | 1 |
| 1 | 1 | 2 |

(If the addition is modulo 2, then the 2 in the crucial corner becomes 0.) We see that the logical operations are identical to the arithmetical ones, except that “ $1+1$ ” must be redefined to be 1, not 2 or 0. Finally, notice that the logical relation of *implication* is equivalent to “less than or equal” in this numerical representation:

| | | |
|-------------------|---|---|
| \longrightarrow | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |

Now recall (or learn, from Appendix 2) that multiplication of matrices is defined by

$$(MN)_{ik} = \sum_j M_{ij}N_{jk}$$

(i.e., each element of the product is calculated by taking the “dot product” of a row of M with a column of N ; the number of columns of M must be equal to the number of rows of N for this to make sense). Therefore:

Definition and Theorem: If \mathcal{R}_1 is a relation from A to B with matrix M_1 and \mathcal{R}_2 is a relation from B to C with matrix M_2 , then $\mathcal{R}_1 \circ \mathcal{R}_2$ is the relation from A to C defined by:

$$a (\mathcal{R}_1 \circ \mathcal{R}_2) c \quad \text{means} \quad \exists b \in B [a \mathcal{R}_1 b \wedge b \mathcal{R}_2 c].$$

The matrix representing $\mathcal{R}_1 \circ \mathcal{R}_2$ is $M_1 M_2$, calculated with the logical addition rule, $1 + 1 = 1$. (That is, “+” actually means “ \vee ” (and “ \times ” means “ \wedge ”). Note that this logical matrix multiplication is easy to carry out: As soon as you find a term 1, you can stop adding up the sum for that element.)

PROOF:

$$a (\mathcal{R}_1 \circ \mathcal{R}_2) c \longleftrightarrow (a \mathcal{R}_1 b_1 \wedge b_1 \mathcal{R}_2 c) \vee (a \mathcal{R}_1 b_2 \wedge b_2 \mathcal{R}_2 c) \vee \cdots,$$

where all elements b_j of B are considered as possible intermediate elements. When translated into the matrix notation, this is precisely the matrix multiplication law.

In a situation (like the example \mathcal{R} above) where no element is related to itself, this theorem can be thought of in terms of electrical switches. Let $A = B = C$ and $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}$. Then the matrix of $\mathcal{R} \circ \mathcal{R}$ is M^2 (that is, MM). The question of whether $1 (\mathcal{R} \circ \mathcal{R}) 4$ is the question of whether, within \mathcal{R} , it is possible to move from 1 to 4 in exactly 2 “jumps”. In A there are two possible 2-step paths from 1 to 4, namely, $1 \rightarrow 2 \rightarrow 4$ and $1 \rightarrow 3 \rightarrow 4$. These paths are “in parallel” in the jargon of electricians: if there is a wire along *either* of them, current can flow from 1 to 4. But the two segments of each path are “in series”: *both* segments (such as $1 \rightarrow 2$ and $2 \rightarrow 4$) must be wired in order for current to flow along that path. We need to check which of these potential “wires” actually exist in the relation \mathcal{R} under study; in the example, we easily see that current can flow from 1 to 4 by way of 3 but not by way of 2, and that is enough to conclude that $1 (\mathcal{R} \circ \mathcal{R}) 4$.

WARNINGS: For historical reasons, the notation for functions is inconsistent with that for relations. If \mathcal{R}_1 and \mathcal{R}_2 are functions, then the relation $\mathcal{R}_1 \circ \mathcal{R}_2$ is the same thing as the function $\mathcal{R}_2 \circ \mathcal{R}_1$. (Recall that in function composition, the function on the left is the one that acts *last*.) Linear algebra students should note that this discrepancy carries over to the matrix representations: When linear functions are represented by matrices, the matrix for the “outer” function (the one

applied last) stands on the *left* of the matrix of the “inner” function in the matrix product; but in a relational matrix product the matrix of the “second” relation stands on the *right*. In linear algebra, the rows of a matrix representing a function are associated with the codomain and the columns with the domain; but when one of our relations is a function ($A \rightarrow B$), the rows of our relational matrix go with the domain (A) and the columns with the codomain (B). Fortunately, however, the rule for multiplying matrices is the same in both situations (rows on the left times columns on the right). Note also that Grimaldi uses a bold-face 1 for a matrix consisting *entirely* of 1s, whereas in linear algebra 1 usually means a square matrix with 1s on the main diagonal and 0s everywhere off-diagonal (the identity matrix). Finally, in linear algebra $M \leq N$ means that $\forall \vec{v} [\vec{v} \cdot M \vec{v} \leq \vec{v} \cdot N \vec{v}]$, which is equivalent to $N - M$ being diagonalizable with all its eigenvalues nonnegative; this is not the same as the definition we’re coming to next:

Definition and Theorem: $M \leq N$ means $\forall j \forall k [M_{jk} \leq N_{jk}]$ Then \mathcal{R} is transitive if and only if its matrix M satisfies $M^2 \leq M$. (This follows from our previous observation about the equivalence of \leq and \longrightarrow .)

In our example,

$$M^2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, the number in the upper right corner confirms our earlier conclusion that $1 (\mathcal{R} \circ \mathcal{R}) 4$.

The other basic properties of relations are easy to recognize from M itself: \mathcal{R} is reflexive iff all the diagonal elements of M are 1. \mathcal{R} is symmetric iff the matrix is symmetric ($M_{jk} = M_{kj}$).

An even more visually graspable (but hard to type!) representation of a relation (from a set A to itself) is a *directed graph*. We draw a dot for each element of A , and an arrow from a_1 to a_2 whenever $a_1 \mathcal{R} a_2$. The relation is reflexive iff every point has a loop attached; it is symmetric if the arrows always go both ways; it is transitive if two points connected by a chain of arrows (all in the same direction) are always also connected directly by a single arrow (in that direction).

An important application is the *precedence graph* for statements in a computer program, where $a \mathcal{R} b$ means that a must be executed before b . If statement c uses a variable defined in b , and b uses a variable defined in a , then it is obvious that $b \mathcal{R} c$ and $a \mathcal{R} b$. Since the precedence relation is transitive, it follows that $a \mathcal{R} c$. Such additional relationships could be sketched in while inspecting a graph drawn to represent the more obvious elementary precedence relationships.