Math. 302 (Fulling) 27 September 2002

Test A – Solutions

Name: ____________________ Number: ____________
(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. *(14 pts.)*

   (a) What is the numerical value of $\binom{7}{4}$?

$$\frac{7!}{4!\,(7-4)!} = \frac{7\cdot 6\cdot 5}{3!} = 7\cdot 5 = 35.$$

   (b) Show (by any method) that $\displaystyle\sum_{k=0}^{7}\binom{7}{k} = 2^7$.

Method 1: $\binom{7}{k}$ is the number of k-element subsets of a set of 7 elements (in other words, the number of choices of k things out of 7 things). So $\sum_{k=0}^{7}\binom{7}{k}$ is the number of *all* subsets of a 7-element set, which equals 2^7 (because each element has the choice of being in the subset or not).

Method 2: Use the binomial theorem.

$$2^7 = (1+1)^7 = \sum_{k=0}^{7}\binom{7}{k}1^k 1^{7-k} = \sum_{k=0}^{7}\binom{7}{k}.$$

2. *(14 pts.)* We know that $\neg(p\wedge q) \iff (\neg p \vee \neg q)$ (one of De Morgan's laws). Prove by induction the corresponding law for arbitrarily many propositions:

$$\neg(p_1\wedge p_2\wedge\cdots\wedge p_n) \iff (\neg p_1\vee\neg p_2\vee\cdots\vee\neg p_n) \qquad \text{for } n = 2,3,\ldots,\infty\,.$$

Base step: For $n=2$, the statement is the ordinary De Morgan law, which we are given.

Inductive step: Assume the statement for n . Then for $n+1$, using De Morgan again with $p\mapsto p_1\wedge\cdots p_n$ and $q\mapsto p_{n+1}$, we have

$$\begin{aligned}\neg(p_1\wedge p_2\wedge\cdots\wedge p_n\wedge p_{n+1}) &\iff \neg(p_1\wedge p_2\wedge\cdots\wedge p_n)\vee\neg p_{n+1}\\ &\iff (\neg p_1\vee\neg p_2\vee\cdots\vee\neg p_n)\vee\neg p_{n+1}\\ &\iff \neg p_1\vee\neg p_2\vee\cdots\vee\neg p_n\vee\neg p_{n+1}\,.\end{aligned}$$

In Questions 3 and 4 it is permissible to leave the answers in terms of factorials.

3. *(28 pts.)* John owns 30 books — 15 on computer science, 10 on mathematics, and 5 on physics.

(a) In how many ways can he choose 3 computer books and 2 noncomputer books to take with him to Padre Island?

$$\binom{15}{3}\binom{10+5}{2} = \frac{15!}{3!\,12!}\,\frac{15!}{2!\,13!}\,.$$

(b) In how many ways can he choose 3 books, of which at least 2 are on computer science?

Case 1: 2 CS books. $\binom{15}{2}\binom{15}{1} = \frac{15!\,15}{2!\,13!}$.

Case 2: 3 CS books. $\binom{15}{3} = \frac{15!}{3!\,12!}$.

Total: $\frac{15!\,15}{2!\,13!} + \frac{15!}{3!\,12!}$, which (for curiosity) works out as

$$\frac{3\cdot 15\cdot 15! + 13\cdot 15!}{13!\,3!} = 15\cdot 14(45+13)/6 = 5\cdot 7\cdot 58 = 2030.$$

Remark: The answer $\binom{15}{2}\binom{28}{1}$ is wrong, because it triple-counts the cases where all three books are CS.

(c) In how many ways can he line up all his books on a shelf so that no two computer science books are together?

Each pair of CS books must be separated by another book. The last book can go at the beginning or the end, or along with another non-CS book at one of the 14 interior spaces, for 16 choices in all. Given any such pattern, there are $15!$ ways of ordering each set of books. So the answer is $16(15!)^2$, or $16!\,15!$.

Alternative argument: There are $15!$ ways to order the non-CS books. Then there are $P(16,15) = 16!$ ways to fit the CS books into the 16 available slots (14 internal and 2 at the ends).

(d) In how many ways can he divide the books into 3 piles (without regard to subject matter)? Empty piles are allowed. Piles are *indistinguishable*.

This question should not have been on this test. It requires the methods of Sec. 5.3.

Remark: The answer $\binom{32}{30}$ or $\binom{32}{2}$ is for *distinguishable* piles of *indistinguishable* books. We need the reverse.

4. *(12 pts.)* Consider the expansion of $(a+b+c+d)^8$. (Don't write it out!)

(a) After the expression is simplified (by the commutative law), how many terms will there be?

Put 8 things (factors) into 4 boxes (variables): $\binom{8+4-1}{8} = \frac{11!}{8!\,3!}$.

(b) What will be the coefficient of $a^3b^2c^3$?

The multinomial coefficient $\dfrac{8!}{3!\,2!\,3!\,0!}$.

5. *(18 pts.)* Construct the truth table for $[p \to (q \to r)] \to [p \to r]$. $(*)$
What conclusions can you draw about this proposition?

p	q	r	$q \to r$	$p \to (q \to r)$	$p \to r$	$*$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	0	1	1	1
0	1	1	1	1	1	1
1	0	0	1	1	0	0
1	0	1	1	1	1	1
1	1	0	0	0	0	1
1	1	1	1	1	1	1

Conclusions: The proposition is not a tautology. It is false precisely in the case $p \wedge \neg q \wedge \neg r$. (Thus the proposition is equivalent to $\neg p \vee q \vee r$.)

6. *(14 pts.)* Every student in my class is either a senior or an engineering major. Every oceanography major in the class is married to another student in the class. There is a student in the class who is a senior and is married to an engineering major in the class.

(a) Which of these sentence structures is represented by the foregoing list of facts?

(A) $\forall x[s(x) \vee e(x)] \wedge \forall x[o(x) \to \exists y\, m(x,y)] \wedge \exists x\{s(x) \wedge \exists y[e(y) \wedge m(x,y)]\}$

(B) $\forall x[s(x) \vee e(x)] \wedge \exists y \forall x[o(x) \to m(x,y)] \wedge \exists x\{s(x) \vee \exists y[e(y) \wedge m(x,y)]\}$

(C) $\forall x[s(x) \wedge e(x)] \vee \forall x[o(x) \vee \exists y\, m(x,y)] \vee \exists x\{s(x) \wedge \forall y[e(y) \vee m(x,y)]\}$

(A)

(b) Make up a sentence or paragraph to match one of the other choices. (Be sure to indicate which pattern your "story" matches.) It is understood that the class of students is the universe of discourse.

(B) Every student is either a senior or an engineering major. There is a student who is married to every oceanography major in the class. There is a student who either is a senior, or is married to an engineering major.

(C) Either every student is a senior engineering major; or every student is either an oceanography major or married to someone in the class; or there is a senior who is married to every student who is not an engineering major. [In the last clause I used $e(y) \vee m(x,y) \iff \neg e(y) \to m(x,y)$ to avoid a very awkward sentence construction.]