Math. 302 (Fulling) 25 October 2002

Test B – Solutions

Name: ______________________________ Number: ______________

(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* Declare each of these inferences valid or invalid. Thoroughly justify your answers (by counterexamples, formal deductions, or Venn diagrams with commentary).

(a) $\exists x\,[p(x) \vee q(x)] \;\Rightarrow\; \exists x\, p(x) \vee \exists x\, q(x)$

VALID. Here is a deduction in the Quine style:

1. $p(c) \;\rightarrow\; \exists x\, p(x)$ (existential generalization)
2. $q(c) \;\rightarrow\; \exists x\, q(x)$ (existential generalization)
3. $*\;\; \exists x\,[p(x) \vee q(x)]$ (hypothesis)
4. $*\;\; p(c) \vee q(c)$ (letting c be such an x) [FLAG c]
5. $*\;\; \exists x\, p(x) \vee \exists x\, q(x)$ (by (1) and (2) — step called "constructive dilemma" by Grimaldi)
6. $\exists x\,[p(x) \vee q(x)] \;\rightarrow\; \exists x\, p(x) \vee \exists x\, q(x)$

Here is the same argument in words: By hypothesis, there is some object that satisfies at least one of the two conditions. So, at least one of the two conditions is satisfied by something! In Venn diagram terms, a dot inside the union of two regions must be in one region or the other. (And vice versa — so (a) remains valid if the arrow is reversed.)

And here is a slightly simpler formal deduction (gleaned from several student papers):

1. $*\;\; \exists x\,[p(x) \vee q(x)]$ (hypothesis)
2. $*\;\; p(c) \vee q(c)$ (letting c be such an x) [FLAG c]
3. $*\;\; p(c) \vee \exists x\, q(x)$ (valid because $q \;\rightarrow\; r \;\Rightarrow\; p \vee q \lim p \vee r$)
4. $*\;\; \exists x\, p(x) \vee \exists x\, q(x)$ (same reason)
5. $\exists x\,[p(x) \vee q(x)] \;\rightarrow\; \exists x\, p(x) \vee \exists x\, q(x)$

(b) $\forall x\,[p(x) \vee q(x)] \;\Rightarrow\; \forall x\, p(x) \vee \forall x\, q(x)$

INVALID. Counterexample: Every integer is either even or odd, but it is not true that every integer is even, nor that every integer is odd.

(c) $\forall x\, p(x) \vee \forall x\, q(x) \;\Rightarrow\; \forall x\,[p(x) \vee q(x)]$

VALID. Since $p \;\rightarrow\; p \vee q$ is a tautology, it is clear that $\forall x\, p(x) \;\rightarrow\; \forall x\,[p(x) \vee q(x)]$, and similarly for $\forall x\, q(x)$. So under either of those hypotheses, the conclusion follows. (This could be made into a formal deduction like (a), with the flag occurring at the final step of universal generalization.)

In Venn terms, if all the points are inside one region or the other, certainly they are all in the union. (But that argument does not go the other way, hence (b) was invalid.)

2. *(14 pts.)* Prove that $n^4 \le 4^n$ for all $n \ge 4$.

The claim is certainly true for $n = 4$. We can use that as the base of an induction.

Assume that the claim is true for n . Look at

$$\begin{aligned}(n+1)^4 &= n^4 + 4n^3 + 6n^2 + 4n + 1 \\ &\le 2n^4 + 6n^2 + n^2 + 1 \quad (\text{since } 4 \le n) \\ &\le 2n^4 + 8n^2 \le 2n^4 + 2n^3 \\ &\le 4n^4 \le 4 \cdot 4^n \quad (\text{by inductive hypothesis}) \\ &= 4^{n+1} \quad (\text{Q.E.D.})\end{aligned}$$

3. *(15 pts.)* Prove: $A \times (B \cup C) = (A \times B) \cup (A \times C)$. *Suggestion:* Start with the hypothesis $(a, x) \in A \times (B \cup C)$ and apply various set-theory definitions and logic laws.

$$\begin{aligned}
(a, x) \in A \times (B \cup C) &\iff (a \in A) \wedge (x \in B \cup C) \\
&\iff a \in A \wedge (x \in B \vee x \in C) \\
&\iff (a \in A \wedge x \in B) \vee (a \in A \wedge x \in C) \\
&\iff [(a, x) \in A \times B] \vee [(a, x) \in A \times C] \\
&\iff (a, x) \in [(A \times B) \cup (A \times C)]
\end{aligned}$$

4. *(28 pts.)* In mini-poker the deck has 3 suits of 5 cards each (A-2-3-4-5), and a hand consists of 3 cards. What is the probability of drawing each of these hands? (Please leave the answers as fractions, not decimals.)

The question [parts (c)–(e)] is ambiguous because it was not made clear whether the ace can be either high or low in a straight (as in standard poker) or only low. I will concentrate on the case "ace always low" but state the "high-low" results as afterthoughts.

First observe that the total number of hands is

$$\binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{6} = 5 \cdot 7 \cdot 13 = 455.$$

This will be the denominator in each probability calculation.

(a) three of a kind (for example, three aces)

There is only one such hand for each rank, or 5 in all.

$$\text{Probability} = \frac{5}{5 \cdot 7 \cdot 13} = \frac{1}{91}\,.$$

(b) a pair (but not three of a kind)

We have $5 \cdot \binom{3}{2}$ [ranks $\times$ suits] choices for the pair, and then $4 \cdot \binom{3}{1}$ choices for the extra card, for a total of $20 \cdot 9 = 180$.

$$\text{Probability} = \frac{2^2 \cdot 3^2 \cdot 5}{5 \cdot 7 \cdot 13} = \frac{36}{91}\,.$$

(c) a straight flush

There are 3 suits, and for each suit there are 3 choices for the lowest card (since the highest straight in this game is 3-4-5). So there are 9 straight flushes.

$$\text{Probability} = \frac{9}{455} \quad \text{(already in lowest terms).}$$

[If the 4-5-A straight is also allowed, there are 4 choices for low card, hence 12 straight flushes and $P = 12/455$.]

(d) a flush (all cards of the same suit, but not a straight flush)

There are 3 suits and $\binom{5}{3}$ flushes of each suit, of which 3 are straight flushes:

$$3\left[\binom{5}{3}-3\right]=3[5\cdot 4/2-3]=3\cdot 7=21.$$

$$\text{Probability}=\frac{3\cdot 7}{5\cdot 7\cdot 13}=\frac{3}{65}\,.$$

[In the ace-high case there are only 18 flushes that are not straight, so $P=18/455$. Interestingly, in this case straight flushes are not particularly rare compared to other flushes.]

(e) a straight (three cards in a row, but not a straight flush)

There are 3 choices for the lowest card (as we already saw) and 3 choices of suit for each of the 3 cards, yielding $3\cdot 3^3=81$. But we must subtract the 9 straight flushes, getting 72.

$$\text{Probability}=\frac{72}{455}$$

(already in lowest terms, since the prime factors of 72 are all twos and threes). [When aces can be high, the calculation runs $4\cdot 3^3-12=96$, so $P=96/455$.]

(f) cards of three different suits

The first card could be anything; then there are 10 choices for the second, and 5 choices for the third. But the order in which these cards came up doesn't count, so we have overcounted by a factor of 3! permutations.

$$\binom{15}{1}\binom{10}{1}\binom{5}{1}\frac{1}{3!}=\frac{15}{3}\cdot\frac{10}{2}\cdot 5=5^3=125.$$

$$\text{Probability}=\frac{5^3}{5\cdot 7\cdot 13}=\frac{25}{91}\,.$$

Simpler argument (found on one student paper): For each suit, there are $\binom{5}{1}$ choices of the card representing that suit, so the number of such hands is 5^3.

Checks:

(1) The number of hands with no pairs is

$$3^3\ [\text{suits}]\ \cdot\binom{5}{1}\binom{4}{1}\binom{3}{1}\ [\text{ranks}]\ \cdot\frac{1}{3!}\ [\text{permutations}]=27\cdot 5\cdot 2=270.$$

Add the pairs and three-of-a-kind hands: $270+180+5=455$. $\surd$

(2) The number of hands with exactly two cards of the same suit is

$$3\binom{5}{2}\cdot 2\binom{5}{1}=6\cdot\frac{5!}{2!\,3!}\cdot 5=5^2\cdot 4\cdot 3=300.$$

Add the flushes (including straight flushes) and the hands with all different suits: $300+(21+9)+125=455$. $\surd$

5. *(14 pts.)* Prove by induction: $\left(\dfrac{d}{dx}\right)^n e^{x^2} = e^{x^2} P_n(x)$, where $P_n(x)$ is a polynomial of degree n containing only even powers of x if n is even and only odd powers of x if n is odd.

Base: For $n = 0$ the assertion is $e^{x^2} = e^{x^2} P_0(x)$, or $P_0(x) = 1$, which is indeed a polynomial of degree 0 containing only even powers.

Induction: Assume the theorem for n and consider the derivative of order $n+1$. It is

$$\frac{d}{dx}\left[e^{x^2} P_n(x)\right] = e^{x^2}\left[2xP_n(x) + P'(x)\right].$$

By the inductive hypotheses about P_n, the function in the brackets is a polynomial of degree $n+1$ whose exponents differ from those of P_n by ± 1, so both the differentiation and the multiplication by x change even powers to odd and vice versa. Therefore, P_{n+1} has the asserted properties.

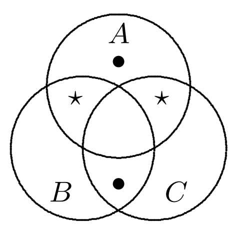
Remark: With extra work (or by specializing a gargantuan formula for iterating the chain rule called *Faà di Bruno's theorem*) one can show that

$$\left(\frac{d}{dx}\right)^n e^{x^2} = e^{x^2} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!\, 2^{n-2m}}{m!\,(n-2m)!}\, x^{n-2m}.$$

Here $\lfloor \frac{n}{2} \rfloor$ is the smallest integer less than or equal to $\frac{n}{2}$.

6. *(14 pts.)* Prove or disprove: $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$. (Recall that $\triangle$ is the *symmetric difference*: $K \triangle L = (K - L) \cup (L - K)$.)

The assertion is false.



The four marked regions make up $A \triangle (B \cap C)$. The two regions marked "$\bullet$" make up $(A \triangle B) \cap (A \triangle C)$. The two regions marked "$\star$" are in $A \triangle (B \cap C)$ but not in $(A \triangle B) \cap (A \triangle C)$. Thus $(A \triangle B) \cap (A \triangle C) \subset A \triangle (B \cap C)$ but the reverse inclusion is false.

Here is a more deductive argument: If $x \in (A \triangle B) \cap (A \triangle C)$, then x is in either A or B but not both, and x is in either A or C but not both. So either x is in A *only* (i.e., in the upper bulleted region of the diagram), or x is in both B and C but not A (i.e, in the lower bulleted region). Thus x is in either A or $B \cap C$ but not both. This proves the inclusion. On the other hand, if x is in $A \cap B$ but not in C (i.e., in the left-side starred region), then x is in A but not in $B \cap C$, so $x \in A \triangle (B \cap C)$; but as we just saw, such an x is not in $(A \triangle B) \cap (A \triangle C)$, because it is in both A and B. (The analogous remark applies to the other starred region.) This shows that the sets are not equal.