Math. 302 (Fulling) 22 November 2002

Test C – Solutions

Name: ______________________ Number: ____________
(as on attendance sheets)

Calculators may be used for simple arithmetic operations only!

1. *(25 pts.)* [Use part (a) to answer part (b).]

   (a) What is the sequence whose generating function is $f(x) = \dfrac{x^4}{1-x}$?

The Maclaurin series of f is

$$x^4 \sum_{n=0}^{\infty} x^n = \sum_{j=4}^{\infty} x^j = x^4 + x^5 + \cdots .$$

So the sequence of coefficients is

$$\{0,0,0,0,1,1,1,1,1,\ldots\} .$$

   (b) In how many ways can 20 robots be assigned to 3 assembly lines with at least 4 robots on each line?

Part (a) is the generating function for number of robots assigned to one line. The generating function for the problem at hand is the cube of that:

$$\begin{aligned}\left(\frac{x^4}{1-x}\right)^3 &= \frac{x^{12}}{(1-x)^3} \\ &= x^{12} \sum_{j=0}^{\infty} \binom{-3}{j} (-x)^j \\ &= x^{12} \sum_{j=0}^{\infty} \binom{3+j-1}{j} (+x)^j\end{aligned}$$

by the generalized binomial theorem (whose j th coefficient can be constructed as $(j!)^{-1}(-3)(-3-1)\cdots$). The coefficient in the final series can also be written as

$$\binom{3+j-1}{3-1} = \binom{2+j}{2} .$$

We need to extract the coefficient of x^{20} — i.e., the one with $j=8$:

$$\binom{10}{8} = \binom{10}{2} = \frac{10 \times 9}{2} = 45.$$

Remarks: (1) For the generating-function method to apply, it is understood that the robots are *indistinguishable* and the lines are distinguishable. This problem is essentially the same as Exercise 10 in Sec. 9.2. (2) The sum in the generating function f extends to infinity; there is no need to truncate it at $j = 20$. Doing so, however, will not (by itself) cause an error, because there is no way for exponents greater than 20 to contribute to the exponent 20 in the f^3 series.

Alternative arguments (useful to check the answer): (1) Assign 4 robots to each line. Then distribute the remaining 8 in all possible ways: Put 8 indistinguishable objects into 3 boxes,

$$\binom{8+(3-1)}{8} = 45.$$

(2) Start out by brute force: If line A has 4 robots, the possibilities are

$$\begin{array}{ccc} 4 & 4 & 12 \\ 4 & 5 & 11 \\ \vdots & & \\ 4 & 12 & 4 \end{array}$$

There are $12 - 4 + 1 = 9$ such cases. Similarly, if line A has 5 robots, the most line C can have is 11, and there are 8 such cases. And so on, ... till when line A has 12 robots, there is only 1 case. The grand total is

$$\sum_{i=1}^{9} i = \frac{9(9+1)}{2} = 45.$$

2. *(24 pts.)* Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Count these:

(a) the Cartesian product, $A \times B$. [How many elements are in it?]

$$|A||B| = 4 \times 3 = 12.$$

(b) functions from A to B

$$|B|^{|A|} = 3^4 = 81.$$

(c) relations from A to B

$$2^{12} \; (= 4096) \quad \text{(from (a))}.$$

(d) injective (one-to-one) functions from A to B

0. (A is too big to fit into B in a one-to-one way.)

(e) injective functions from B to A

$$P(4,3) = 4 \times 3 \times 2 = \frac{4!}{(4-3)!} = 24.$$

(f) surjective (onto) functions from B to A

0. (B is too small to fill up A.)

3. *(25 pts.)* Consider these functions:

$$f_1(n) = n! + n^2, \qquad f_2(n) = n^3 + 2n^2 - 3, \qquad f_3(n) = n^2 \ln n + n^5 \ln(\ln n),$$

$$f_4(n) = ne^n(\pi + n^4), \qquad f_5(n) = e^{2n} \ln n + e^n \ln(n^2).$$

(a) For each function f_j in the list, give a *simple*, standard function g_j such that $f_j \in \Theta(g_j)$.

$$g_1(n) = n! \quad \text{or } \sqrt{n}\left(\frac{n}{e}\right)^n$$
$$g_2(n) = n^3$$
$$g_3(n) = n^5 \ln(\ln n)$$
$$g_4(n) = n^5 e^n$$
$$g_5(n) = e^{2n} \ln n$$

(b) Arrange the functions f_j in increasing order of growth at infinity. [Answer should look like $f_a \prec f_b \prec \cdots$, where $f \prec h$ means that $f \in O(h)$ but $h \notin O(f)$.]

The ordering is the same as that of the corresponding g functions.

$$f_2 \prec f_3 \prec f_4 \prec f_5 \prec f_1.$$

4. *(26 pts.)* Show that one of these matrices represents an *equivalence relation* (on the set $\{1, 2, 3\}$), and explain why each of the others does not.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

An equivalence relation must be reflexive, symmetric, and transitive. Let's investigate these three properties in turn.

Reflexive: In terms of the matrix, this condition is $\forall j\, M_{jj} = 1$; that is, all entries on the main diagonal (upper left to lower right) are 1. Thus B is not reflexive, but the other four are. (To avoid awkward language, I don't make a distinction between a relation and the matrix that represents it.)

Symmetric: This condition is $M_{jk} = M_{kj}$; that is, elements located symmetrically with respect to the main diagonal are equal. By inspection, C and D are not symmetric, but the other three are.

Transitive: This is harder to see just by looking at the matrices. One can either draw the corresponding directed graph (possibly easier when working with pencil and paper) or calculate the Boolean square of the matrix and check the property $M^2 \leq M$ (much easier to type!).

$$A^2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = A,$$

$$B^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} > B,$$

$$C^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} > C,$$

$$D^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = D,$$

$$E^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} > E.$$

Thus A and D are transitive, but B , C , and E are not.

The verdicts:

A is an equivalence relation.
B is neither reflexive nor transitive.
C is neither symmetric nor transitive.
D is not symmetric.
E is not transitive.