Math. 302 (Fulling) 22 September 2004

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* By constructing a truth table, determine whether or not

$$(p \wedge \neg q) \;\rightarrow\; \neg(p \;\longleftrightarrow\; q)$$

is a tautology.

p	q	p	$\wedge$	$\neg$	q	$\rightarrow$	$\neg$	p	$\leftrightarrow$	q
0	0	0	0	1	0	1	0	0	1	0
0	1	0	0	0	1	1	1	0	0	1
1	0	1	1	1	0	1	1	1	0	0
1	1	1	0	0	1	1	0	1	1	1

Since the column resulting for the principal connective, $\rightarrow$, is entirely 1s, the statement framework is a tautology.

2. *(10 pts.)* Let $S = \{\text{Bush}, \text{Kerry}\}$, and let $T = \{\text{Alabama}, \text{Alaska} \ldots\}$ be the 50 United States.

(a) How many elements does the Cartesian product $S \times T$ have?

$2 \times 50 = 100$.

(b) Which of the following is a subset of $S \times T$?

(A) $\{(\text{Bush}, \text{Texas})\}$

YES. (It is a subset containing only one element.)

(B) $\{(\text{California}, \text{Kerry}), (\text{Texas}, \text{Bush})\}$

NO. (Each ordered pair is in the wrong order — this belongs in $T \times S$.)

(C) $\{\text{Bush}, \text{Delaware}, \text{Louisiana}\}$

NO. (The elements are not ordered pairs. This belongs in $S \cup T$.)

(D) all of these

(E) none of these

3. *(20 pts.)* Let $F(x,y)$ mean “ x can fool y ”, where the universe of discourse consists of all people. Use quantifiers to express each of these statements.

(a) Everyone can be fooled by somebody.

$\forall y\, \exists x\, F(x,y)$.

(b) Nobody can fool everybody.

$\neg \exists x\, \forall y\, F(x,y)$.

(c) There are exactly two people who can fool everybody.

$\exists x\, \exists z\, \{x \neq z \wedge \forall y\, [F(x,y) \wedge F(z,y)] \wedge \forall w\, [(w \neq x \wedge w \neq z) \;\rightarrow\; \neg \forall y\, F(w,y)]\}$.

(undoubtedly other correct answers are possible.)

(d) Everybody can fool Fred.

$\forall x\, F(x, \text{Fred})$.

4. *(25 pts.)* Express the logical structure of the sentences in the given paragraph in terms of propositional variables (p , q , ...). Then determine whether the argument is valid.

   Either Beth is not enrolled as a student, or she is exempt from social security tax. If an employee is exempt from social security tax, then [s]he does not get a pay stub every week. To get an income tax refund, Beth must show that she is enrolled as a student and must submit copies of her weekly pay stubs. Therefore, Beth cannot get the income tax refund.

Let

$$\begin{aligned}
s &\iff \text{Beth is a student,} \\
e &\iff \text{Beth is exempt from social security tax,} \\
p &\iff \text{Beth has pay stubs,} \\
r &\iff \text{Beth gets the tax refund.}
\end{aligned}$$

In order, the three premises have the structures

$$\begin{aligned}
&\neg s \vee e\,, \\
&e \;\rightarrow\; \neg p\,, \\
&r \;\rightarrow\; (s \wedge p) \quad \text{(the two conditions being \textit{necessary}, maybe not sufficient),}
\end{aligned}$$

and the alleged conclusion is simply $\neg r$. (In this simple, informal argument, we need not distinguish between submitting the stubs and merely receiving them, etc. Some students pointed out that the premises as stated do not lead to a conclusion if Beth is not an employee; technically that objection is correct, although common sense indicates that in that case there would be no pay stubs and no income tax to be refunded.) The first premise can be rewritten as $s \rightarrow e$, which by syllogism with the second premise yields $s \rightarrow \neg p$. This last is equivalent to $\neg s \vee \neg p$, which is equivalent by a De Morgan law to $\neg(s \wedge p)$. So, by the contrapositive of the third premise (in other words, by Modus Tollens), r must be false. Therefore, the argument is valid.

5. *(15 pts.)* Recall that $A - B$ is the set of those elements that are in A but not in B . Prove that

$$A - B = \overline{B \cup \overline{A}}\,.$$

Start by simplifying the right side by De Morgan to $\overline{B} \cap A$. Now write out the definitions of membership in each side:

$$\begin{aligned}
x \in A - B &\iff x \in A \wedge x \notin B, \\
x \in \overline{B} \cap A &\iff x \notin B \wedge x \in A.
\end{aligned}$$

These are the same!

6. *(15 pts.)* One of these logical equivalences is correct and one is incorrect. Determine which one is incorrect and explain why. For extra credit, prove the correct one. (*Note:* q is a proposition that does not contain the letter x.)

(A) $\{[\forall x\, p(x)] \rightarrow q\} \iff \forall x\,[p(x) \rightarrow q]$

This one is wrong. Suppose that q is false and $p(x)$ is true sometimes but not always. Then the left side is $\mathbf{F} \rightarrow \mathbf{F}$, or $\mathbf{T}$, while the right side is falsified by letting x equal any c for which $p(c)$ is true.

Alternative argument: Transform the left side:

$$[\forall x\, p(x)] \rightarrow q \iff [\neg\forall x\, p(x)] \vee q \iff [\exists x\, \neg p(x)] \vee q.$$

On the other hand, the right side is equivalent to $\forall x\,[\neg p(x) \vee q]$, and it is easy to see that that is equivalent to $[\forall x\, \neg p(x)] \vee q$. (Think separately about the cases q true and q false, or construct a formal deduction like the one in (B).) But certainly $[\exists x\, \neg p(x)] \vee q$ and $[\forall x\, \neg p(x)] \vee q$ are not the same.

(B) $\{q \rightarrow [\forall x\, p(x)]\} \iff \forall x\,[q \rightarrow p(x)]$

Let's try to deduce the right side from the left:

$q \rightarrow [\forall x\, p(x)]$
$*$ Assume q
$*$ $\forall x\, p(x)$
$*$ $p(c)$ for an arbitrary c
$q \rightarrow p(c)$
$\forall x\,[q \rightarrow p(x)]$ since c was arbitrary. $[c]$

Now the other way:

$\forall x\,[q \rightarrow p(x)]$
$q \rightarrow p(c)$ for an arbitrary c
$*$ Assume q
$*$ $p(c)$
$*$ $\forall x\, p(x)$ since c was arbitrary. $[c]$
$q \rightarrow \forall x\, p(x)$.