Math. 302 (Fulling) 20 October 2004

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(10 pts.)* Let A and B be sets, and f a function from A into B ($f\colon A \to B$).

   (a) Explain what it means for f to be "one-to-one".

f does not map two different elements of A into the same element of B. (There are many equivalent ways of saying this. However, beware of formulations that interchange the roles of A and B; they merely say that f is a function!)

   (b) If $|A| = 3$ and $|B| = 10$, how many one-to-one functions $f\colon A \to B$ are there? ($|A| =$ number of elements in A, etc.)

$10 \cdot 9 \cdot 8 = 720$.

2. *(20 pts.)* Find a formula (or set of formulas) for the nth derivative of $f(x) = \sin(\pi x)$, and prove it by mathematical induction ($n \in \mathbf{N}$).

Let's write out the first few cases:

$$\begin{aligned}
f^{(0)}(x) &\equiv f(x) = \sin(\pi x), \\
f^{(1)}(x) &\equiv f'(x) = \pi \cos(\pi x), \\
f^{(2)}(x) &= -\pi^2 \sin(\pi x), \\
f^{(3)}(x) &= -\pi^3 \cos(\pi x), \\
f^{(4)}(x) &= \pi^4 \sin(\pi x), \\
f^{(5)}(x) &= \pi^5 \cos(\pi x), \quad \ldots
\end{aligned}$$

It is clear that the pattern will repeat. Thus for n even we will get $\pi^n \sin(\pi x)$ times a sign, and for n odd, $\pi^n \cos(\pi x)$ times a sign. After some experimentation and checking, if necessary, we get a way to represent the sign, and we write the formula

$$f^{(n)}(x) = \begin{cases} (-1)^{n/2} \pi^n \sin(\pi x) & \text{if } n \text{ is even,} \\ (-1)^{(n-1)/2} \pi^n \cos(\pi x) & \text{if } n \text{ is odd.} \end{cases}$$

Alternative formula (found on two student papers):

$$f^{(n)}(x) = \pi^n \sin\left(\pi x + \frac{n\pi}{2}\right).$$

Now to the proof:
Base: We already took care of that in our exploratory calculations.
Induction: If n is even, we assume the formula for the preceding odd number,

$$f^{(n-1)}(x) = (-1)^{(n-2)/2} \pi^{n-1} \cos(\pi x).$$

Differentiating gives

$$f^{(n)}(x) = (-1)^{n/2} \pi^n \sin(\pi x)$$

as required. Similarly, differentiating the even formula yields the odd formula. (To prove the alternative formula, use $\cos z = \sin(z + \pi/2)$.)

3. *(15 pts.)* For each function, find the simplest function in the same Θ class.

(a) $(2n^2 + 3n - 5)(3\sqrt{n} + n)$

n^3. (Find the fastest growing product and discard the numerical coefficient.)

(b) $n^3 \ln n + ne^n - 5n^2 e^{n/2}$

ne^n. (It will beat $n^p e^{n/2}$ for any p. If in doubt, take the limit:

$$\frac{ne^n}{n^2 e^{n/2}} = \frac{e^{n/2}}{n} \to \infty .)$$

(c) $10n! - \lfloor n^2 \rfloor$

$n!$. (There is a complicated way to write $\Theta(n!)$ in terms of exponentials and powers (Stirling's formula), but $n!$ appears so often that we usually accept it as a "simple" function in itself. Note that $n!$ is $O(n^n)$ but *not* $\Theta(n^n)$; n^n grows faster!)

4. *(20 pts.)* The automatic teller machine of the Last National Bank of Old Dime Box uses three-character passwords consisting of letters and digits. (There is no distinction between upper- and lower-case letters.) A password must contain at least one digit. The first character must be a letter. How many possible passwords are there?

There are 26 choices for the initial letter. For the second and third characters there are these cases:

$$\begin{array}{ll} 10 \cdot 10 & \text{digit} + \text{digit}, \\ 10 \cdot 26 & \text{digit} + \text{letter}, \\ 26 \cdot 10 & \text{letter} + \text{digit}, \end{array}$$

which add up to 620. Alternatively, we can calculate the total number of pairs and subtract the pairs with no digits:

$$36^2 - 26^2 = 1296 - 676 = 620.$$

In any event, the answer is

$$26 \cdot 620 = 16,120.$$

5. *(15 pts.)* Use l'Hôpital's rule and mathematical induction to show that $(\ln x)^n \in O(x)$ for all $n \in \mathbf{Z}^+$.

Base: $\ln x \in O(x)$, so it's true for $n = 1$. (In fact, it's also true for $n = 0$.)
Induction: Use the limit theorem to compare $(\ln x)^n$ with x:

$$\lim_{x\to\infty} \frac{(\ln x)^n}{x} = \lim_{x\to\infty} \frac{n(\ln x)^{n-1}\frac{1}{x}}{1} = n \lim_{x\to\infty} \frac{(\ln x)^{n-1}}{x}.$$

So if the limit is 0 for $n - 1$, then it is also 0 for n. (The limit in the base case is

$$\lim_{x\to\infty} \frac{(\ln x)^0}{x} = 0.)$$

6. *(20 pts.)* **Open-book.** Exercise 44, p. 272. (Note that the definitions of "leaves" and "internal vertices" appear *above* the exercise.)

We need to show that $l(T) = i(T) + 1$ for every full binary tree, where l is the number of leaves and i the number of internal vertices. Binary trees are defined in Definition 6 and illustrated in Figure 4. (Note that "full" does not mean "balanced"; thus the number of leaves is not necessarily of the form 2^n .)

Base step: The tree consisting of a single vertex has $l = 1$ and $i = 0$, so it satisfies the equation.

Induction: Assume $l(T_1) = i(T_1) + 1$ and $l(T_2) = i(T_2) + 1$ and study the tree $T_1 \cdot T_2$. Its leaves are just the union of the leaves of the two parts: $l(T_1 \cdot T_2) = l(T_1) + l(T_2)$. Its internal vertices are those of the parts, plus the root vertex added at the top (see Def. 6), so $i(T_1 \cdot T_2) = i(T_1) + i(T_2) + 1$. Thus (by the inductive assumption) $l(T_1 \cdot T_2) = i(T_1) + i(T_2) + 2 = i(T_1 \cdot T_2) + 1$.

Alternative induction (found on several student papers): From Fig. 4 it is obvious that big trees are built up from smaller trees by attaching two new leaves to an old leaf, which thereby becomes an internal vertex — no longer a leaf. Therefore, at each such step the number of leaves grows by $2 - 1 = 1$ and the number of internal vertices grows by 1 . Thus the relation $l = i + 1$ is preserved.