Math. 302 (Fulling) 17 November 2004

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(36 pts.)*

(a) Count the distinct arrangements (permutations) of the letters in AARDVARK. (All letters are used, order matters, but different letters A (for instance) are indistinguishable.)

Count the permutations of all the letters, then divide to correct for the overcounting of those that are indistinguishable:

$$\frac{8!}{3!\,2!\,1!\,1!\,1!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{6\cdot 2} = 56\cdot 60 = 3360\,.$$

(b) How many such arrangements have no consecutive As?

Count the arrangements of the other letters, remembering to divide by $2!$ to avoid overcounting the Rs. These 5 letters leave 6 possible positions for the As, of which we must choose 3. So the total number of possibilities is

$$\frac{5!}{2!}\,\frac{6!}{3!\,3!} = (5\cdot 4\cdot 3)(5\cdot 4) = 1200\,.$$

Alternative method: Count the arrangements with adjacent As and subtract.

$$\text{All As together:}\quad \frac{6!}{2!} = 360$$

$$\text{As grouped 2 and 1:}\quad \frac{7!}{2!} = 7\cdot 360$$

$$3360 - 8\cdot 360 = 480\,.$$

Oops! We should not have subtracted the cases where the pair of As ends up adjacent to the single A; that gave us two more copies of the 360. Adding back 720 to 480 gives 1200, as expected.

(c) How many ways are there to choose 4 letters from AARDVARK so that no letter appears more than twice? (Order doesn't matter.)

The generating function for the possible number of As is $1+x+x^2$, and the same is true for R. For each of the other 3 letters the generating function is $1+x$. The complete generating function is

$$\begin{aligned}(1+x+x^2)^2(1+x)^3 &= (1+2x+x^2+2x^2+2x^3+x^4)(1+3x+3x^2+x^3)\\ &= \text{irrelevant terms} + x^4(2\cdot 1+3\cdot 3+2\cdot 3+1\cdot 1)\\ &= \cdots + 18x^4\,.\end{aligned}$$

So the answer is 18.

Alternative method: In poker terminology, there are

$$\text{hands with two pairs:}\quad 1$$

$$\text{hands with one pair:}\quad 2\cdot\binom{4}{2} = 12$$

$$\text{hands with no pair:}\quad \binom{5}{4} = 5$$

Adding gives 18.

2. *(25 pts.)* Solve $a_n - 4a_{n-2} = \frac{1}{9}\,n3^n\,, \quad a_0 = 0\,, \quad a_1 = 0\,.$

First solve the homogeneous relation, $a_n - 4a_{n-2} = 0$. Try $a_n = r^n$, getting $r^n - 4r^{n-2} = 0$, or $r^2 = 4$. Thus $r = \pm 2$, and the general homogeneous solution is

$$A\,2^n + B\,(-2)^n\,.$$

Now find a particular solution of the nonhomogeneous relation in the form $a_n = (Cn + D)3^n$. We get

$$\begin{aligned}\frac{1}{9}\,n3^n &= (Cn + D)3^n - 4[C(n-2) + D]3^{n-2}\\ &= n3^n\left[C - \frac{4}{9}C\right] + 3^n\left[D - \frac{4}{9}(-2C + D)\right]\\ &= \frac{5}{9}\,C\,n3^n + \left(\frac{8}{9}\,C + \frac{5}{9}\,D\right)3^n.\end{aligned}$$

Therefore,

$$5C = 1\,, \qquad 8C + 5D = 0\,,$$

so

$$C = \frac{1}{5}\,, \qquad D = -\,\frac{8C}{5} = -\,\frac{8}{25}\,.$$

Thus the general solution is

$$a_n = \frac{1}{5}\,n3^n - \frac{8}{25}\,3^n + A\,2^n + B\,(-2)^n\,.$$

The initial conditions give

$$0 = -\frac{8}{25} + A + B\,, \qquad 0 = \frac{3}{5} - \frac{24}{25} + 2A - 2B\,.$$

or

$$A + B = \frac{8}{25}\,, \qquad A - B = \frac{9}{50}\,.$$

Therefore,

$$\begin{aligned}A &= \frac{1}{2}\left(\frac{8}{25} + \frac{9}{50}\right) = \frac{25}{100} = \frac{1}{4}\,,\\ B &= \frac{1}{2}\left(\frac{8}{25} - \frac{9}{50}\right) = \frac{7}{100}\,.\end{aligned}$$

Finally, after one final simplification,

$$a_n = \frac{1}{5}\,n3^n - \frac{8}{25}\,3^n + 2^{n-2} + \frac{7}{25}\,(-2)^{n-2}\,.$$

3. *(10 pts.)* Solve $a_{n+1} = (n+3)a_n\,, \quad a_0 = 1\,.$

Let's write out the first few terms.

$$a_0 = 1\,, \qquad a_1 = 3\,, \qquad a_2 = 4\cdot 3\,, \ldots.$$

It is clear that

$$a_n = \frac{(n+2)!}{2}\,.$$

4. *(14 pts.)* Estimate the growth of $f(n)$ if

$$f(n) = 24f(n/5) + 100n^2 \ln n\,.$$

(Use of the "log" button on your calculator is permitted but shouldn't really be necessary.) Explain your reasoning clearly.

$\log_5 24$ is slightly less than 2, because $\log_5 25$ would be 2. Therefore, it seems that Case 3 should apply. However, we need to check the extra condition in that case.

$$\begin{aligned} 24 \cdot 100 \left(\frac{n}{5}\right)^2 \ln\left(\frac{n}{5}\right) &= \frac{24}{25} \cdot 100n^2(\ln n - \ln 5) \\ &< \frac{24}{25} \cdot 100n^2 \ln n. \end{aligned}$$

Since $c = 24/25 < 1$, the condition is satisfied. Conclusion:

$$f(n) \in \Theta(n^2 \ln n).$$

(Since the factor 100 does not affect the truth of a Θ statement, we can drop it.)

5. *(15 pts.)* By the method of generating functions, count the solutions of

$$x + y + z = 12 \qquad \text{with} \qquad 0 \le x \le 3\,, \quad 5 \le y \le 6\,, \quad 2 \le z \le 6\,.$$

$$\begin{aligned} &(1 + x + x^2 + x^3)(x^5 + x^6)(x^2 + x^3 + x^4 + x^5 + x^6) \\ &\qquad = (x^5 + 2x^6 + 2x^7 + 2x^8 + x^9)(x^2 + x^3 + x^4 + x^5 + x^6) \\ &\qquad = \text{irrelevant terms} + x^{12}(2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 1 \cdot 1) \\ &\qquad = \cdots + 7x^{12}\,. \end{aligned}$$

So the answer is 7.

Remark: Looking back at where the x^{12} terms came from, we can actually list the solutions:

x	y	z
0	6	6
1	5	6
1	6	5
2	5	5
2	6	4
3	5	4
3	6	3