Solution to Leon-Colley Exercise 5.4.25

Show that, for any $\mathbf{u}$ and $\mathbf{v}$ in a normed vector space,

$$\|\mathbf{u}+\mathbf{v}\| \geq \big|\|\mathbf{u}\| - \|\mathbf{v}\|\big|.$$

We know that, for any $\mathbf{x}$ and $\mathbf{y}$,

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Let $\mathbf{x} = -\mathbf{u}$ and $\mathbf{y} = \mathbf{v}+\mathbf{u}$. Then

$$\|\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}+\mathbf{u}\|,$$

or

$$\|\mathbf{v}+\mathbf{u}\| \geq \|\mathbf{v}\| - \|\mathbf{u}\|.$$

Obviously the situation is symmetrical in $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$, so

$$\|\mathbf{u}+\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|.$$

So altogether,

$$\|\mathbf{u}+\mathbf{v}\| \geq \big|\|\mathbf{u}\| - \|\mathbf{v}\|\big|.$$