Math. 311 (Fulling) 7 December 2012

Final Examination – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* $\mathbf{A}$ is a vector field, f is a scalar function. Label each of these identities as either (i) nonsense (gibberish), (ii) meaningful but false, or (iii) correct.

(a) $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f\nabla \times \mathbf{A}$

Correct. To check this it suffices to look at one component:

$$[\nabla \times (f\mathbf{A})]_z = \frac{\partial}{\partial x}(fA_y) - \frac{\partial}{\partial y}(fA_x) = \frac{\partial f}{\partial x}\,A_y - \frac{\partial f}{\partial y}\,A_x + f\,\frac{\partial A_y}{\partial x} - f\,\frac{\partial A_x}{\partial y} = [\nabla f \times \mathbf{A}]_z + f[\nabla \times \mathbf{A}]_z\,.$$

(b) $\nabla \times (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f\nabla \cdot \mathbf{A}$

Nonsense: the left side is a vector but the right side is a scalar.

(c) $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} - f\nabla \times \mathbf{A}$

Meaningful but false (by comparison with (a)).

2. *(Essay – 20 pts.)* Prove that for any linear function L, the kernel of L is a subspace of the domain of L. (Along the way it would be a good idea to define "subspace" and "kernel".

The kernel is the set of vectors satisfying $L\vec{v} = 0$. We note that $L(\lambda\vec{v} + \vec{u}) = \lambda L\vec{v} + Lv = 0$ (by definition of linearity), when $\vec{v}$ and $\vec{u}$ are any vectors in the kernel and λ is any scalar. This shows that the kernel is closed under addition and scalar multiplication, so (by definition) it is a subspace.

3. *(30 pts.)* Find an **ORTHONORMAL** basis of eigenvectors of $A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$.

The characteristic equation is

$$0 = \det(A-\lambda) = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{vmatrix} = (2-\lambda)[(3-\lambda)^2-1] = (2-\lambda)[\lambda^2-6\lambda+8] = -(\lambda-2)^2(\lambda-4).$$

(Resist the temptation to multiply out the cubic; we *want* to factor the polynomial, not expand it.)

$\lambda = 2$ *(double root):*

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \;\Rightarrow\; \begin{cases} x = z, \\ y \text{ arbitrary}, \\ z \text{ arbitrary}. \end{cases}$$

Therefore, an ON basis for this subspace is

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\lambda = 4$ *(single root):*

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{cases} x = -z, \\ y = 0, \\ z \text{ arbitrary.} \end{cases}$$

Therefore, a normalized eigenvector for this one-dimensional subspace is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

All three vectors go together to form an ON eigenbasis for all of $\mathbf{R}^3$.

4. *(10 pts.)* A function $f(x, y, z)$ satisfies $f(0,0,0) = 10$, $\nabla f(0,0,0) = 0$, and

$$\left\{ \frac{\partial^2 f}{\partial x_j \, \partial x_k} \right\} = A \quad \text{(where } A \text{ is the matrix in the previous problem).}$$

(a) Is $(0,0,0)$ a maximum, minimum, or saddle point of f?

A mininum, since all 3 eigenvalues are positive.

(b) What is the best affine approximation to f at $(0,0,0)$?

The tangent plane is horizontal, since $\nabla f = 0$. The best approximation is simply the constant function, $f(x) \approx 10$.

5. *(30 pts.)*

(a) Find a basis of eigenvectors of $M = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$.

The characteristic equation is

$$0 = \begin{vmatrix} 2-\lambda & 1 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda + 1) - 4 = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2).$$

Therefore, the eigenvalues are 3 and -2.

$\lambda = 3$: $\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x = y$. So an eigenvector is $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = -2$: $\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow y = -4x$. So an eigenvector is $\vec{v}_2 = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

Of course, the vectors should *not* be orthogonal, because the matrix M is not symmetric.

(b) Solve the ODE system $\begin{cases} \dfrac{dx}{dt} = 2x + y\,, & x(0) = x_0\,, \\ \dfrac{dy}{dt} = 4x - y\,, & y(0) = y_0\,. \end{cases}$

Method 1: Let $\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. Then it has the form

$$\vec{r}(t) = c_1 \vec{v}_1 e^{3t} + c_2 \vec{v}_2 e^{-2t}$$

for some constants c_1 , c_2 that must be chosen to match the initial conditions:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - 4c_2 \end{pmatrix}.$$

Row-reduce:

$$\begin{pmatrix} 1 & 1 & x_0 \\ 1 & -4 & y_0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & x_0 \\ 0 & -5 & y_0 - x_0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{4}{5} x_0 + \frac{1}{5} y_0 \\ 0 & 1 & \frac{1}{5}(x_0 - y_0) \end{pmatrix}.$$

So

$$c_1 = \frac{4}{5} x_0 + \frac{1}{5} y_0\,, \quad c_2 = \frac{1}{5}(x_0 - y_0)\,.$$

Method 2: Let $D = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}$, $U^{-1} = \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix}$. Check that $M = UDU^{-1}$. Now the solution matrix is

$$Ue^{tD}U^{-1} = \frac{1}{5} \begin{pmatrix} 4e^3 t + e^{-2t} & e^{3t} - e^{-2t} \\ 4e^{3t} - 4e^{-2t} & e^{3t} + 4e^{-2t} \end{pmatrix}.$$

Apply it to the initial data vector:

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= U \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \frac{x_0}{5} \begin{pmatrix} 4e^{3t} + e^{-2t} \\ 4e^{3t} - 4e^{-2t} \end{pmatrix} + \frac{y_0}{5} \begin{pmatrix} e^{3t} - e^{-2t} \\ e^{3t} + 4e^{-2t} \end{pmatrix} = \frac{4x_0 + y_0}{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + \frac{x_0 - y_0}{5} \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-2t} \end{aligned}$$

in agreement with the other method.

6. *(30 pts.)* Let $\mathbf{F} = 3x\,\hat{\imath} + 2y\,\hat{\jmath} + z^2\,\hat{k}$.

(a) Calculate $\nabla \cdot \mathbf{F}$.

$$3 + 2 + 2z = 2z + 5$$

(b) Calculate $\nabla \times \mathbf{F}$.

$$(0 - 0)\vec{\imath} + \cdots = 0 \qquad \text{(since each } F_j \text{ depends only on } x_j\text{)}$$

(c) Calculate $\iint_P \mathbf{F}\cdot d\mathbf{S}$ when P is the portion of the diagonal plane with equation $x = z$ that lies inside the unit cube ($0 < x, y, z < 1$).

As clarified during the test, the normal vector is the upward one ($n_z > 0$). We take x and y as independent variables with $z = x$.

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = dx\,.$$

$$\begin{aligned}\iint_P \bigl(F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy\bigr) &= \int_0^1\!\int_0^1 3x\,(-dx\,dy) + \int_0^1\!\int_0^1 2y\,(dx)^2 + \int_0^1\!\int_0^1 x^2\,dx\,dy \\ &= \int_0^1\!\int_0^1 (x^2 - 3x)\,dx\,dy = \frac{1}{3} - \frac{3}{2} = -\frac{7}{6}\,.\end{aligned}$$

Remark: The negative sign on the first term makes geometrical sense, because the "upward" side of the surface is leaning to the left ($n_x < 0$).

(d) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$ when C is the curve $x = t$, $y = \cos t$, $z = \sin t$ ($0 < t < \pi$).

Because the curl is 0 , we may choose any path between $(0,1,0)$ and $(\pi,-1,0)$. So I take $z = 0$ and let $t = y$ run from 1 to -1 with $x = 0$, then let $t = x$ run from 0 to π with $y = -1$.

$$\int \mathbf{F}(\mathbf{r})\cdot \mathbf{r}'(t)\,dt = \int_1^{-1} 2y\,dy + \int_0^{\pi} 3x\,dx = 0 + \frac{3\pi^2}{2}\,.$$

This problem can also be solved by constructing a potential function. Also, it is not too hard to evaluate the integral directly, along the original path.

7. *(10 pts.)* For whichever of these sets is not linearly independent, find an independent set with the same span.

(a) $\{e^x, e^{-x}, \sinh x, \cosh x\}$

Any two from the list will do.

(b) $\{e^x, xe^{2x}, x^2e^{3x}\}$

independent

8. *(25 pts.)*

(a) Find all solutions of $\begin{cases} x + 2y - z = 10\,, \\ x + y + 2z = 20\,. \end{cases}$

$$\begin{pmatrix} 1 & 2 & -1 & 10 \\ 1 & 1 & 2 & 20 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 20 \\ 1 & 2 & -1 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 20 \\ 0 & 1 & -3 & -10 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 5 & 30 \\ 0 & 1 & -3 & -10 \end{pmatrix}.$$

Thus z is arbitrary and

$$y = -10 + 3z\,, \qquad x = 30 - 5z\,.$$

(b) Discuss the rank, kernel, and range of the linear function with matrix $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix}$.

The rank is 2, since the rows are independent. Therefore, the range is all of $\mathbf{R}^2$. Therefore, the kernel has dimension $3-2=1$. In fact, the kernel is the z-dependent part of the solution found in (a); a basis vector for the kernel is $(-5,3,1)^{\mathrm{t}}$.

9. *(30 pts.)* An inner product is defined by the formula

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt\,.$$

Find its first 3 orthogonal polynomials. (Don't bother to normalize the last one.)

$\langle 1,1\rangle = \int_0^1 1\,dt = 1$, so $p_0 = 1$ is already normalized.

$\langle 1,t\rangle = \int_0^1 t\,dt = \frac{1}{2}$. Therefore, $p_{1\perp} \equiv t - \frac{1}{2}$ is orthogonal to p_0. We need to normalize it:

$$\int_0^1 {p_{1\perp}}^2\,dt = \int_0^1 \left(t^2 - t + \frac{1}{4}\right) dt = \left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{t}{4}\right]_0^1 = \frac{1}{12}\,.$$

Therefore,

$$p_1 = \sqrt{12}\left(t - \tfrac{1}{2}\right).$$

$$\langle 1,t^2\rangle = \int_0^1 t^2\,dt = \frac{1}{3}\,.$$

$$\langle p_1,t^2\rangle = \sqrt{12}\int_0^1 \left(t^3 - \frac{t^2}{2}\right) dt = \sqrt{12}\left(\frac{1}{4} - \frac{1}{6}\right) - \frac{1}{\sqrt{12}}\,.$$

So $p_{2\parallel} = \frac{1}{3} + \frac{\sqrt{12}}{\sqrt{12}}\left(t - \frac{1}{2}\right) = t - \frac{1}{6}$, and

$$p_{2\perp} = t^2 - t + \tfrac{1}{6}\,.$$