Math. 311 (Fulling) 28 September 2012

Test A – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(10 pts.)* Find a parametric representation of the plane through the points $(1,3,9)$, $(2,2,2)$, and $(0,2,5)$.

First find two vectors parallel to the plane:

$$\vec{u}_1 = (1,3,9) - (2,2,2) = (-1,1,7)\,, \qquad \vec{u}_2 = (0,2,5) - (2,2,2) = (-2,0,3)\,.$$

Then the general point on the plane is

$$\vec{r} = (2,2,2) + s(-1,1,7) + t(-2,0,3).$$

(This is one representation of the plane among many possible ones.)

2. *(18 pts.)* Solve these systems (introducing parameters when the answer is not unique).

(a) $\begin{pmatrix} 3 & 8 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$.

Reduce the augmented matrix:

$$\begin{pmatrix} 3 & 5 & 6 \\ 1 & 3 & -1 \end{pmatrix} \xrightarrow{(1)\leftrightarrow(2)} \begin{pmatrix} 1 & 3 & -1 \\ 3 & 8 & 6 \end{pmatrix} \xrightarrow{(2)\to(2)-3(1)} \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 9 \end{pmatrix} \xrightarrow[(2)\to-(2)]{(1)\to(1)+3(2)} \begin{pmatrix} 1 & 0 & 26 \\ 0 & 1 & -9 \end{pmatrix}.$$

Therefore, $x = 26$ and $y = -9$ is the only solution.

(b) $\begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

I will not bother to augment the matrix, since the last column will always be zeros.

$$\begin{pmatrix} 3 & 9 \\ 1 & 3 \end{pmatrix} \xrightarrow{(1)\to\frac{1}{3}(1)} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \xrightarrow{(2)\to(2)-(1)} \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the system is equivalent to $x + 3y = 0$. Thus, if $y = s$, then $x = -3s$. The general solution can be written

$$\vec{r} = s \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

3. *(15 pts.)* Construct the best affine approximation (also known as the first-order approximation) to $T(x,y) = \begin{pmatrix} x^3 + 4y^2 \\ \sqrt{x - y^2} \end{pmatrix}$ in the neighborhood of the point $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

The Jacobian matrix is

$$J_{\mathbf{r}_0} = \begin{pmatrix} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} \end{pmatrix} = \left.\begin{pmatrix} 3x^2 & 8y \\ \dfrac{1}{2\sqrt{x-y^2}} & \dfrac{-y}{\sqrt{x-y^2}} \end{pmatrix}\right|_{x_0,y_0} = \begin{pmatrix} 75 & 8 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix}.$$

The approximation then is, in matrix form,

$$T(x,y) \approx T(x_0,y_0) + J_{\mathbf{r}_0}(\mathbf{r} - \mathbf{r}_0) = \begin{pmatrix} 129 \\ 2 \end{pmatrix} + \begin{pmatrix} 75 & 8 \\ \frac{1}{4} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x - 5 \\ y - 1 \end{pmatrix}.$$

There are many acceptable variations on the notation. In such a problem it is better to leave the answer as a function of $x - x_0$ and $y - y_0$ instead of combining the $\mathbf{r}_0$ terms with the constant term.

4. *(16 pts.)* A curve C in two-dimensional space is specified by the parametric equations

$$x = t^2, \quad y = \sin t.$$

(a) Find the tangent vector to C at the point where $t = \pi$.

$$\vec{T} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \left.\begin{pmatrix} 2t \\ \cos t \end{pmatrix}\right|_{\pi} = \begin{pmatrix} 2\pi \\ -1 \end{pmatrix}.$$

(b) Find the directional derivative of $f(x,y) = x + e^y$ at that point, in the direction of the curve.

We need to normalize the tangent vector: $\hat{T} = \dfrac{1}{\sqrt{4\pi^2+1}}\vec{T}$.

$$\frac{df}{d\hat{T}} = \nabla f \cdot \hat{T} = (1 \quad e^y)|_{t=\pi} \begin{pmatrix} 2\pi \\ -1 \end{pmatrix} \frac{1}{\sqrt{4\pi^2+1}}.$$

At the point in question, $y = 0$ (and $x = \pi^2$, although we don't need that). Therefore,

$$\frac{df}{d\hat{T}} = (1 \quad 1) \begin{pmatrix} 2\pi \\ -1 \end{pmatrix} \frac{1}{\sqrt{4\pi^2+1}} = \frac{2\pi - 1}{\sqrt{4\pi^2+1}}.$$

5. *(9 pts.)* When u is a function of a (real) variable x, define $A(u)$ to be a new function of x given by the formula $A(u)(x) = u(x) + \int_0^x t^2 u(t)\,dt$. Prove or disprove that A is a linear operator.

Since each term involves u to the first power. we suspect that A is linear. Let us test it:

$$\begin{aligned} A(ru_1+u_2)(x) &= (ru_1+u_2)(x) + \int_0^x t^2(ru_1+u_2)(t)\,dt \\ &= ru_1(x) + u_2(x) + r\int_0^x t^2 u_1(t)\,dt + \int_0^x u_2(t)\,dt = [rA(u_1)+A(u_2)](x). \end{aligned}$$

6. *(12 pts.)* **Do ONE of these [(A) or (B)].** (Up to 6 points extra credit for doing the other one.)

   (A) Producing a refrigerator requires 0.1 ton of steel and 0.2 ton of plastic. Producing an airplane requires 5 tons of steel and 2 tons of plastic. Producing a ton of steel consumes 3 tons of coal and 10 barrels of water. Producing a ton of plastic consumes 2 tons of coal and 50 barrels of water. Organize these facts into matrices, and find the matrix that tells you how much coal (c) and water (w) is needed to make r refrigerators and a airplanes.

This is the same as Question 4 of Test A of 2002.

Let s and p be the quantities of steel and plastic, and let

$$\begin{pmatrix} s \\ p \end{pmatrix} = B\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix}\begin{pmatrix} r \\ a \end{pmatrix}, \qquad \begin{pmatrix} c \\ w \end{pmatrix} = A\begin{pmatrix} s \\ p \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix}\begin{pmatrix} s \\ p \end{pmatrix}.$$

Then $\begin{pmatrix} c \\ w \end{pmatrix} = AB\begin{pmatrix} r \\ a \end{pmatrix}$, where

$$AB = \begin{pmatrix} 3 & 2 \\ 10 & 50 \end{pmatrix}\begin{pmatrix} 0.1 & 5 \\ 0.2 & 2 \end{pmatrix} = \begin{pmatrix} 0.7 & 19 \\ 11 & 150 \end{pmatrix}.$$

   (B) Calculate $\displaystyle \frac{d}{dx}\int_x^{x^2} \frac{\cos(tx)}{t}\,dt$.

By the multivariable chain rule, there will be three terms, one for each appearance of x. Two of them involve applications of the fundamental theorem of calculus, with a sign change when the lower limit is involved:

$$\frac{\cos(x^3)}{x^2}(2x) - \frac{\cos(x^2)}{x}.$$

The third term involves the derivative of the integrand, and it leaves an integral that can be evaluated:

$$\int_x^{x^2} (-\sin(tx))\,dt = -\frac{1}{x}\int_{x^2}^{x^3} \sin z\,dz = \frac{\cos(x^3)-\cos(x^2)}{x}.$$

Adding, we get

$$\frac{3\cos(x^3) - 2\cos(x^2)}{x}.$$

7. *(20 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1)\leftrightarrow(2)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{(3)\to(3)-2(1)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 & -2 & 1 \end{pmatrix} \xrightarrow{(3)\to(3)-4(2)} \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -4 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow[(1)\to(1)-(3),\ (2)\to(2)-(3)]{(3)\to-(3)/3} \begin{pmatrix} 1 & 0 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{4}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Thus

$$M^{-1} = \frac{1}{3} \begin{pmatrix} -4 & 1 & 1 \\ -1 & -2 & 1 \\ 4 & 2 & -1 \end{pmatrix}.$$

This can easily be checked:

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 3 \end{pmatrix} \begin{pmatrix} -4 & 1 & 1 \\ -1 & -2 & 1 \\ 4 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$