Math. 311 (Fulling) 15 December 2014

Final Examination – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(18 pts.)* Which of these formulas define *inner products* on $\mathbf{R}^3$? Explain what's wrong with those that don't. ($\mathbf{r}_1 = (x_1, y_1, z_1)$, etc.)

(a) $\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = {x_1}^2 {x_2}^2 + {y_1}^2 {y_2}^2 + {z_1}^2 {z_2}^2$

NO — not bilinear.

(b) $\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = x_1 x_2 - y_1 y_2 + z_1 z_2$

NO — not positive definite.

(c) $\langle \mathbf{r}_1, \mathbf{r}_2 \rangle = x_1 x_2 + 2y_1 y_2 + 3z_1 z_2$

YES (bilinear, symmetric, positive definite).

2. *(15 pts.)* Three variables are constrained by the equations

$$x^2 + y^2 + z^2 = 26 \quad \text{and} \quad xyz = -12\,.$$

Find $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$ at the point $(x, y, z) = (1, -3, 4)$.

Differentiate both equations with respect to z :

$$2x\,\frac{\partial x}{\partial z} + 2y\,\frac{\partial y}{\partial z} + 2z = 0 \qquad yz\,\frac{\partial x}{\partial z} + xz\,\frac{\partial y}{\partial z} + xy = 0\,.$$

Plug in the numbers:

$$2\,\frac{\partial x}{\partial z} - 6\,\frac{\partial y}{\partial z} = -8\,, \qquad -12\,\frac{\partial x}{\partial z} + 4\,\frac{\partial y}{\partial z} = 3\,.$$

Solve by either Gauss–Jordan or Cramer method:

$$\frac{\partial x}{\partial z} = \frac{7}{32}\,, \qquad \frac{\partial y}{\partial z} = \frac{45}{32}\,.$$

3. *(35 pts.)* Let $M = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & \sqrt{2} \\ 0 & \sqrt{2} & 3 \end{pmatrix}$.

(a) Find *all* eigenvalues and eigenvectors of M .

The characteristic (or secular) equation is

$$0 = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & 2-\lambda & \sqrt{2} \\ 0 & \sqrt{2} & 3-\lambda \end{vmatrix} = (4-\lambda)[(\lambda-2)(\lambda-3)-2] = (4-\lambda)(\lambda^2 - 5\lambda + 4) = (\lambda-4)^2(\lambda-1)$$

— from which it is clear that the eigenvalues are

$$\lambda = 4 \quad \text{(a double root)}, \qquad \lambda = 1\,.$$

Eigenvectors with $\lambda = 4$ *:* Solve the homogeneous system with matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & \sqrt{2} \\ 0 & \sqrt{2} & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} :$$

$$x \text{ is arbitrary}\,, \qquad y = \frac{z}{\sqrt{2}}\,, \qquad z \text{ is arbitrary}\,.$$

Eigenvectors with $\lambda = 1$ *:* Solve the homogeneous system with matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & \sqrt{2} & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} :$$

$$x = 0\,, \qquad y = -\sqrt{2}\,z\,, \qquad z \text{ is arbitrary}\,.$$

(b) A function $f(x,y,z)$ satisfies $f(0,0,0) = 10$, $\nabla f(0,0,0) = 0$, and

$$\left\{ \frac{\partial^2 f}{\partial x_j\, \partial x_k} \right\} = M\,.$$

Is $(0,0,0)$ a maximum, minimum, or saddle point of f?

minimum, because all eigenvalues are positive.

(c) Find an *orthogonal* matrix U that diagonalizes M (that is, $M = UDU^{-1}$).

This amounts to finding an orthonormal basis of eigenvectors and using them as the columns of U. For $\lambda = 4$, the eigenvector with $z = 0$ and the one with $x = 0$ are already orthogonal, but we need to normalize the second one to unit length. For $\lambda = 1$ we just need to normalize the one eigenvector. Result:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} \end{pmatrix}.$$

4. *(10 pts.)* Prove the formula $\nabla \cdot (f\mathbf{A}) = f\,\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$, or correct the formula if it is wrong.

The best method is brute force:

$$\nabla \cdot (f\mathbf{A}) \equiv \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z) = \frac{\partial f}{\partial x}\,A_x + f\,\frac{\partial A_x}{\partial x} + \text{analogous terms} \equiv \nabla f \cdot \mathbf{A} + f\,\nabla \cdot \mathbf{A}\,.$$

5. *(30 pts.)*

(a) Find a basis of eigenvectors of $M = \begin{pmatrix} -2 & 1 \\ 2 & -3 \end{pmatrix}$.

The characteristic equation is

$$0 = \begin{vmatrix} -2-\lambda & 1 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+2)(\lambda+3) - 2 = \lambda^2 + 5\lambda + 4 = (\lambda+1)(\lambda+4)\,.$$

So the eigenvalues are $\lambda = -1$ and $\lambda = -4$.

Eigenvectors with $\lambda = -1$*:* Matrix

$$\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow -x + y = 0 \Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

will do. (To do this problem we don't need an *orthonormal* basis; in fact, we can't get one, because the two eigenspaces are not orthogonal, since the matrix is not symmetric.)

Eigenvectors with $\lambda = -4$*:* Matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow 2x + y = 0 \Rightarrow \vec{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

will do.

So $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is an eigenbasis.

(b) Solve the ODE system $\begin{cases} \dfrac{dx}{dt} = -2x + y\,, & x(0) = x_0\,, \\ \dfrac{dy}{dt} = 2x - 3y\,, & y(0) = y_0\,. \end{cases}$

The easiest way to organize this calculation is to use each eigenvector in the basis to construct a simple exponential solution, then add them with unknown coefficients:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = A\vec{u}_1 e^{-t} + B\vec{u}_2 e^{-4t}\,.$$

Then set $t = 0$:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} A + B \\ A - 2B \end{pmatrix}.$$

Solve: $\quad A = \frac{1}{3}(2x_0 + y_0)\,, \qquad B = \frac{1}{3}(x_0 - y_0)\,.$

Summary of alternative method: Put the eigenvectors together to get a matrix

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \qquad U^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix},$$

that diagonalizes M and hence splits the problem into two decoupled scalar differential equations. Then the solution matrix is

$$e^{tM} = U \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-4t} \end{pmatrix} U^{-1} = \frac{1}{3}\begin{pmatrix} 2e^{-t} + e^{-4t} & e^{-t} - e^{-4t} \\ 2e^{-t} - 2e^{-4t} & e^{-t} + 2e^{-4t} \end{pmatrix};$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-tM} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

which multiplies out to the same solution as above.

6. *(30 pts.)* Let $\mathbf{F} = x\,\hat{\imath} + y\,\hat{\jmath} + z^2\,\hat{k}$. Evaluate each of these surface integrals by a direct method (that is, *don't* try to use the Gauss or Stokes theorem).

(a) $\displaystyle\iint_{S_1} \mathbf{F}\cdot d\mathbf{S}$, where S_1 is the cylinder $r = 4$, $0 < z < 2$. (Do not include end faces.)

The normal vector to the cylinder is

$$\hat{n} = \hat{r} = \frac{x}{r}\,\hat{\imath} + \frac{y}{r}\,\hat{\jmath} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}.$$

Therefore the integrand is

$$\mathbf{F}\cdot\hat{n} = \frac{x^2}{r} + \frac{y^2}{r} = r = 4\,,$$

and the integral is that times the surface area,

$$(4)(2)(2\pi)(4) = 64\pi\,.$$

(b) $\displaystyle\iint_{S_2} \mathbf{F}\cdot d\mathbf{S}$, where S_2 is the parabolic surface $z = -x^2$, $-1 < x < 1$, $0 < y < 2$.
(Use "mystical rules" or, equivalently, the method of three 2×2 determinants.)

We want to use x and y as the independent variables, so we work out

$$dz = -2x\,dx + 0\,dy$$

and

$$\begin{aligned}
\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} &= \iint F_x\,dy\,dz + \iint F_y\,dz\,dx + \iint F_z\,dx\,dy \\
&= \iint x\,dy\,(-2x\,dx) + \iint y(-2x\,dx)\,dx + \iint z^2\,dx\,dy \\
&= \iint (+2)x^2\,dx\,dy + 0 + \iint x^4\,dx\,dy \\
&= \int_0^2 dy \int_{-1}^{1} (2x^2 + x^4)\,dx \\
&= 2\left[\frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^{1} = 4\left[\frac{2}{3} + \frac{1}{5}\right] = \frac{52}{15}\,.
\end{aligned}$$

7. *(26 pts.)* Determine whether each set is linearly independent. If not, find an independent set with the same span.

(a) $\{(1,0,1,0), (2,1,1,1), (5,1,4,1)\}$

Reduce

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 5 & 1 & 4 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix},$$

and one more step reduces the bottom line to zeros. So the set is DEPENDENT, and a basis for its span is either the two nonzero rows of the final matrix, or any two vectors from the original set.

(b) $\{(1,0,1,0),(2,6,1,1),(5,1,4,2)\}$

Proceeding as before, you find that the bottom row does not zero out this time. So the set is INDEPENDENT.

(c) $\{1,\cos t,\sin t,\cos^2 t,\sin^2 t\}$

DEPENDENT. The identity $\cos^2 t+\sin^2 t = 1$ can be used to eliminate one of the last two elements.

(d) $\{t^2+1, t^2-1, t\}$

INDEPENDENT. The first two elements are a basis for $\operatorname{span}\{t^2,1\}$. Or, if that's not obvious, set up the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and it's clear the rows are independent.

8. *(16 pts.)* Find an *orthonormal* basis for $\operatorname{span}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\}$.

As usual in Gram–Schmidt problems, I'll call the given vectors $\{\vec v_1, \vec v_2\}$. I'll start with the unit vector along $\vec v_1$,

$$\vec u_1 = \frac{\vec v_1}{\|\vec v_1\|} = \frac{1}{\sqrt 6}\begin{pmatrix}1\\2\\1\end{pmatrix}.$$

Then the parallel part of $\vec v_2$ is

$$\vec v_{2\|} = \langle \vec u_1, \vec v_2\rangle \vec u_1 = \frac56 \begin{pmatrix}1\\2\\1\end{pmatrix},$$

so the perpendicular part is

$$\vec v_{2\perp} = \begin{pmatrix}2\\1\\1\end{pmatrix} - \frac56\begin{pmatrix}1\\2\\1\end{pmatrix} = \frac16\begin{pmatrix}7\\-4\\1\end{pmatrix}.$$

The square of its norm is $\frac{1}{36}(49+16+1) = \frac{66}{36}$, so the second element of the normalized basis is

$$\vec u_2 = \frac{1}{\sqrt{66}}\begin{pmatrix}7\\-4\\1\end{pmatrix}.$$

9. *(20 pts.)* Do **ONE** of these (up to 8 points extra credit for doing **TWO**).

(A) Prove the theorem that dim(domain) = dim(kernel) + dim(range) for a linear function with a finite-dimensional domain. *Hint:* Choose or construct bases for the various subspaces involved.

[See the solutions for the Fall 2004 final exam, Question 10(A).]

(B) An inner product is defined by the formula

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)g(t)\, e^{-|t|}\, dt\,.$$

Find its first 3 orthogonal polynomials. (Don't bother to normalize the last one.)
FREE INFORMATION: $\int_0^\infty t^n\, e^{-t}\, dt = n!\,.$

Note first that

$$\int_{-\infty}^{\infty} t^n\, e^{-|t|}\, dt = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2n! & \text{if } n \text{ is even.} \end{cases}$$

Let $v_n = t^n$ and let $\hat{u}_n$ be the resulting orthogonal polynomials (normalized).

Step 0: We have $\|v_0\|^2 = \int_{-\infty}^{\infty} e^{-|t|}\, dt = 2$ and hence

$$\hat{u}_0 = \frac{1}{\sqrt{2}}\,.$$

Step 1: $\langle \hat{u}_0, v_1\rangle = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} te^{-|t|}\, dt = 0$. Therefore, $v_{1\perp} = v_1$. Then

$$\|v_{1\perp}\|^2 = \int_{-\infty}^{\infty} t^2 e^{-|t|}\, dt = 4\,,$$

and

$$\hat{u}_1 = \frac{1}{2}\, t\,.$$

Step 2: $\langle \hat{u}_1, v_2\rangle = 0$. Therefore,

$$v_{2\|} = \langle \hat{u}_0, v_2\rangle \hat{u}_0 = \frac{1}{2} \int_{-\infty}^{\infty} t^2 e^{-|t|}\, dt = 2\,.$$

Thus $v_{2\perp} = t^2 - 2$, and here's where we agreed to stop.