Math. 311 (Fulling) 1 December 2014

Test C – Solutions

Name: ____________________________________

Calculators may be used for simple arithmetic operations only!

1. *(23 pts.)* The formula $\langle p,q\rangle = \int_{-\infty}^{\infty} p(t)q(t)\,e^{-t^2}\,dt$ defines an inner product on the vector space of polynomials. Find the first three of the orthonormal polynomials associated with this inner product. (Apply the Gram–Schmidt algorithm to the power functions.) FREE INFORMATION:

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi}\,, \qquad \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\,, \qquad \int_{-\infty}^{\infty} t^4\,e^{-t^2}\,dt = \frac{3\sqrt{\pi}}{4}\,.$$

Let $v_n = t^n$ and let $\hat{u}_n$ be the resulting orthogonal polynomials (normalized).

Step 0: We have $\|v_0\|^2 = \int_{-\infty}^{\infty} e^{-t^2}\,dt = \pi^{1/2}$ and hence

$$\hat{u}_0 = \pi^{-1/4}\,.$$

Step 1: $\langle \hat{u}_0, v_1\rangle = \pi^{-1/4}\int_{-\infty}^{\infty} te^{-t^2}\,dt = 0$ because the integrand is odd. Therefore, $v_{1\perp} = v_1$. Then

$$\|v_{1\perp}\|^2 = \int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt\,,$$

and

$$\hat{u}_1 = \sqrt{2}\,\pi^{-1/4}\,t\,.$$

Step 2: $\langle \hat{u}_1, v_2\rangle = 0$, again because the integrand is odd. Therefore,

$$v_{2\parallel} = \langle \hat{u}_0, v_2\rangle \hat{u}_0 = \pi^{-1/2}\int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt = \frac{1}{2}\,.$$

Thus $v_{2\perp} = t^2 - \frac{1}{2}$, and

$$\|v_{2\perp}\|^2 = \int_{-\infty}^{\infty}\left(t^4 - t^2 + \frac{1}{4}\right)e^{-t^2}\,dt = \sqrt{\pi}\left(\frac{3}{4} - \frac{1}{2} + \frac{1}{4}\right) = \frac{\sqrt{\pi}}{2}\,.$$

So

$$\hat{u}_2 = \sqrt{2}\,\pi^{-1/4}\left(t^2 - \frac{1}{2}\right)\,.$$

2. *(35 pts.)* Let $\mathbf{A}(\mathbf{r}) = xz\,\hat\imath + yz\,\hat\jmath + z^2\,\hat k$ and $\mathbf{B}(\mathbf{r}) = -y\,\hat\imath + x\,\hat\jmath$. Also, let S be the surface of the unit cube with its bottom face omitted. (That is, S is the union of 5 square faces, one of which is $\{0 < x < 1,\, y = 1,\, 0 < z < 1\}$ and the other 4 are similar, the face that is left out being $\{0 < x < 1,\, 0 < y < 1,\, z = 0\}$.) Finally, define $I = \iint_S \mathbf{B}\cdot d\mathbf{S}$ (with the "upward and outward" orientation).

(a) Calculate $\nabla\cdot\mathbf{A}$, $\nabla\times\mathbf{A}$, $\nabla\cdot\mathbf{B}$, and $\nabla\times\mathbf{B}$.

$$\nabla\cdot\mathbf{A} = z + z + 2z = 4z\,.$$

$$\nabla\times\mathbf{A} = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ xz & yz & z^2 \end{vmatrix} = -y\,\hat\imath + x\,\hat\jmath = \mathbf{B}\,.$$

$$\nabla\cdot\mathbf{B} = 0\,.$$

$$\nabla\times\mathbf{B} = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix} = 2\,\hat k\,.$$

(b) Evaluate I.

Because $\nabla\cdot\mathbf{B} = 0$, we can move the surface to the bottom side of the cube. The normal vector to that surface is $\hat k$. But $\mathbf{B}\cdot\hat k = 0$, so the integral is 0.

(c) Evaluate I again, by a completely different method.

Because $\mathbf{B} = \nabla\times\mathbf{A}$, the integral equals the line integral of $\mathbf{A}$ around the square at the base. But $\mathbf{A} = 0$ everywhere on the plane $z = 0$, so again $I = 0$.

(d) Evaluate I by yet a third, completely different, method.

OK, we have to bite the bullet and evaluate the 5 surface integrals. But they are easy: The integral over the top of the cube is 0 because the normal vector is $\hat k$. On the two sides with normal vector $\pm\hat\imath$ the integrand is $\mp y$; the integrals

$$\int_0^1\int_0^1 (\mp y)\,dy\,dz$$

over the two sides cancel. On the two sides with normal vector $\pm\hat\jmath$ the integrand is $\pm x$; the two integrals cancel for the same reason. So $I = 0$.

3. *(12 pts.)* Using an efficient method, evaluate the determinant $\begin{vmatrix} 1 & 0 & 1 & 0 & 2 \\ 5 & 0 & -1 & 3 & 1 \\ 1 & 3 & 2 & 0 & 9 \\ -1 & 0 & 1 & 2 & -2 \\ 1 & 0 & 1 & 1 & 1 \end{vmatrix}$.

There are many correct methods. Here is a good one: Expand in cofactors of the 2nd column:

$$(-3)\begin{vmatrix} 1 & 1 & 0 & 2 \\ 5 & -1 & 3 & 1 \\ -1 & 1 & 2 & -2 \\ 1 & 1 & 1 & 1 \end{vmatrix}.$$

Subtract multiples of the 1st row to clear out the 1st column:

$$(-3)\begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & -6 & 3 & -9 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}.$$

Expand in cofactors of the 1st column:

$$(-3)\begin{vmatrix} -6 & 3 & -9 \\ 2 & 2 & 0 \\ 0 & 1 & -1 \end{vmatrix}.$$

Extract common factors from two rows:

$$(+18)\begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}.$$

Expand in cofactors of the 3rd row:

$$(-18)\left[\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}\right].$$

Combine by multilinearity in the 2nd column:

$$(-18)\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0\,.$$

4. *(30 pts.)* Define coordinates (u, v) in a certain region of $\mathbf{R}^2$ by $\begin{cases} x = u \sinh v\,, \\ y = u \cosh v\,. \end{cases}$

(a) Find the formulas for the tangent vectors to the coordinate curves (at a generic point (u, v)).

$$\frac{\partial \mathbf{r}}{\partial u} = \begin{pmatrix} \sinh v \\ \cosh v \end{pmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial v} = \begin{pmatrix} u \cosh v \\ u \sinh v \end{pmatrix}.$$

(b) Find the formulas (in terms of u and v) for the normal vectors to the coordinate curves.

Here and in (c) the Jacobian determinant will be useful:

$$\begin{vmatrix} \sinh v & u \cosh v \\ \cosh v & u \sinh v \end{vmatrix} = u \sinh^2 v - u \cosh^2 v = -u\,.$$

The inverse matrix is

$$\frac{1}{-u}\begin{pmatrix} u \sinh v & -u \cosh v \\ -\cosh v & \sinh v \end{pmatrix}.$$

The normal vectors are the rows of this matrix:

$$\nabla u = (-\sinh v, \cosh v)\,, \qquad \nabla v = \left(\frac{\cosh v}{u}, -\frac{\sinh v}{u}\right).$$

Comment: We can check that these satisfy the reciprocity relations

$$\nabla u \cdot \frac{\partial \mathbf{r}}{\partial u} = 1\,, \qquad \nabla u \cdot \frac{\partial \mathbf{r}}{\partial v} = 0\,, \qquad \text{etc.}$$

(This checks that you inverted the matrix correctly.) Each basis by itself is not orthonormal, however.

(c) Find the area of the region bounded by the curves $u = 1$, $u = 2$, $v = 0$, and $v = 2$. (Set up the integral, don't evaluate it.)

Integrate the absolute value of the determinant over the region:

$$\int_1^2 du \int_0^2 dv\, u = 2\, \frac{u^2}{2}\bigg|_1^2 = 3\,.$$

(The evaluation is too trivial to resist.)

(d) What is the "certain region"? (Assume that u is always positive but v ranges over all $\mathbf{R}$.) Sketch (in the x–y plane) the coordinate curve $u = 2$ and the coordinate curve $v = 1$. Also, sketch the four vectors in (a) and (b) at the point where $(u, v) = (1, 0)$. (Graphing calculators are allowed.)

The region covered by these coordinates is that where $|x| < y$ — i.e., the "wedge" above both lines $y = -x$ and $y = x$. A curve of constant u is a hyperbola, and a curve of constant v is a straight line coming out of the origin of the Cartesian coordinates. The indicated point is $(x, y) = (0, 1)$, and the vectors there are

$$\frac{\partial \mathbf{r}}{\partial u} = \nabla u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \frac{\partial \mathbf{r}}{\partial v} = \nabla v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

So at this particular point the two bases *are* orthonormal, and equal. The basis is left-handed, reflecting the negative value of the Jacobian determinant.