Math. 311 (Fulling) 28 September 1998

Test A – Solutions

Name: ______________________ Number: ____________

(as on attendance sheets)

1. *(20 pts.)* Find all solutions of the linear system $\begin{cases} 2w + 4x - 2y + 2z = 4, \\ w + 2x + \ y - \ z = 1. \end{cases}$

The augmented matrix is $\begin{pmatrix} 2 & 4 & -2 & 2 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}$.

STRATEGY 1: Interchange rows to bring a 1 to the top.

$$\begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 2 & 4 & -2 & 2 & 4 \end{pmatrix} \xrightarrow{(2)\leftarrow(2)-2(1)} \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & -4 & 4 & 2 \end{pmatrix} \xrightarrow{(2)\leftarrow-\frac{1}{4}(2)} \begin{pmatrix} 1 & 2 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & \frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{(1)\leftarrow(1)-(2)} \begin{pmatrix} 1 & 2 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \end{pmatrix}.$$

STRATEGY 2: Divide top row by 2.

$$\begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{(2)\leftarrow(2)-(1)} \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 2 & -2 & -1 \end{pmatrix} \xrightarrow{(2)\leftarrow\frac{1}{2}(2)} \begin{pmatrix} 1 & 2 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & -\frac{1}{2} \end{pmatrix}$$

$$\xrightarrow{(1)\leftarrow(1)+(2)} \begin{pmatrix} 1 & 2 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \end{pmatrix}.$$

In any case, the equivalent reduced system of equations is

$$w + 2x = \frac{3}{2}\,, \qquad y - z = -\frac{1}{2}\,.$$

We must treat two of the variables as parameters and solve for the other two in terms of those. The standard approach is to start at the end of the list of unknowns and work backwards: Let $z = t$. Then y is determined. Let $x = s$. Solve for y and w:

$$w = \frac{3}{2} - 2s, \qquad y = -\frac{1}{2} + t.$$

2. *(15 pts.)* Colonel Rapidrudder is working out the budget for Rapid Deployment Forces for a meeting of the Joint Chiefs of Staff. A bomber carries a crew of 6 and requires 5000 gallons of fuel. A tank is operated by 3 servicemen and requires 200 gallons of fuel. The Joint Chiefs will be debating how many RDFs to support. It is understood that a Type A Force consists of 10 bombers and 30 tanks, and a Type B Force consists of 5 bombers and 10 tanks. However, the number of Forces of each type is still the subject of debate.

   Find the matrix that Roger R. can use to calculate the personnel and fuel requirements quickly for any proposed scenario. Explain your reasoning clearly.

Let A and B be the number of Forces of each type; let b and t be the numbers of bombers and tanks; let p be the number of persons and f be the gallons of fuel required. Then

$$\begin{pmatrix} b \\ t \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 30 & 10 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}, \qquad \begin{pmatrix} p \\ f \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 5000 & 200 \end{pmatrix} \begin{pmatrix} b \\ t \end{pmatrix}.$$

Thus $\binom{p}{f} = M \binom{A}{B}$ where

$$M = \begin{pmatrix} 6 & 3 \\ 5000 & 200 \end{pmatrix} \begin{pmatrix} 10 & 5 \\ 30 & 10 \end{pmatrix} = \begin{pmatrix} 150 & 60 \\ 56000 & 27000 \end{pmatrix}.$$

3. *(20 pts.)* Let $\vec{F}(\vec{x}) \equiv \begin{pmatrix} x^2 - y^2 + z \\ x^2 + y^2 - z \\ x + y + z \end{pmatrix}$ and let $\vec{x}_0 \equiv \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

   (a) Construct the "best affine approximation" to the vector-valued function $\vec{F}$ in the neighborhood of the point $\vec{x}_0$. (Algebraic simplification is not required.)

$$\vec{F}(\vec{x}_0) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}. \qquad \left.\frac{d\vec{F}}{d\vec{x}}\right|_{\vec{x}_0} = \left.\begin{pmatrix} 2x & -2y & 1 \\ 2x & 2y & -1 \\ 1 & 1 & 1 \end{pmatrix}\right|_{\vec{x}_0} = \begin{pmatrix} 2 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore, the best affine approximation $\hat{F}(\vec{x}) \equiv \vec{F}(\vec{x}_0) + \left.\frac{d\vec{F}}{d\vec{x}}\right|_{\vec{x}_0} (\vec{x} - \vec{x}_0)$ is

$$\hat{F}(\vec{x}) = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \\ z \end{pmatrix}.$$

   (b) Find $d\vec{F}/dt$ at $t = 0$, if $x = t + 1$, $y = 1 - t^2$, $z = \sin t$.

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \vec{x}_0. \qquad \left.\frac{d\vec{x}}{dt}\right|_{t=0} = \left.\begin{pmatrix} 1 \\ -2t \\ \cos t \end{pmatrix}\right|_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, using the chain rule $\frac{d\vec{F}}{dt} = \frac{d\vec{F}}{d\vec{x}} \frac{d\vec{x}}{dt}$ and an intermediate result of (a), we get

$$\left.\frac{d\vec{F}}{dt}\right|_{t=0} = \begin{pmatrix} 2 & -2 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

4. *(15 pts.)* Define A by $[Ay](t) \equiv \dfrac{d^2y}{dt^2} + t^2 y$. Thus A is a mapping from $\mathcal{C}^2(-\infty,\infty)$ into $\mathcal{C}(-\infty,\infty)$. Is A linear or nonlinear? Justify your answer.

Let y_1 and y_2 be two functions, and r a number. Then $A(ry_1+y_2) = \frac{d^2}{dt^2}(ry_1+y_2) + t^2(ry_1+y_2)$. By familiar properties of differentiation and multiplication, this is

$$ry_1'' + y_2'' + rt^2y_1 + t^2y_2 = r(y_1'' + t^2y_1) + (y_2'' + t^2y_2) = r\,Ay_1 + Ay_2\,.$$

This proves that A is linear.

5. *(10 pts.)* Calculate $\frac{d}{dt}\int_1^{t^2} \frac{\ln(tx)}{x}\,dx$.

We use the chain rule and the first fundamental theorem. The chain rule provides one term for each occurrence of t in the expression:

$$\frac{\ln(t^3)}{t^2}\,\frac{d}{dt}t^2 + \int_1^{t^2} \frac{1}{tx}\,x\,\frac{dx}{x}\,.$$

(Remarks: (1) There is no term corresponding to the lower limit of the integral, because that limit does not depend on t . If it did, the associated chain rule factor would be the derivative of the lower limit (not $2t$). (2) The structure of the second term comes from applying the ordinary single-variable chain rule to the logarithm function.) Simplify:

$$\frac{\ln(t^3)}{t^2}\,(2t) + \int_1^{t^2}\frac{dx}{tx} = \frac{2\ln(t^3)}{t} + \frac{1}{t}\,\ln x\Big|_1^{t^2} = \frac{6\ln t}{t} + \frac{1}{t}\,\ln(t^2) = \frac{8\ln t}{t}\,.$$

6. *(20 pts.)* Find the inverse of the matrix $M = \begin{pmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 3 \end{pmatrix}$.

Reduce an augmented matrix:

$$\begin{pmatrix} 0 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{(1)\leftrightarrow(2)} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 0 \\ 2 & -2 & 3 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[(2)\leftarrow\frac12(2)]{(3)\leftarrow(3)-2(1)} \begin{pmatrix} 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac12 & \frac12 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{(1)\leftarrow(1)+(2)} \begin{pmatrix} 1 & 0 & \frac32 & \frac12 & 1 & 0 \\ 0 & 1 & \frac12 & \frac12 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix} \xrightarrow[(2)\leftarrow(2)-\frac12(3)]{(1)\leftarrow(1)-\frac32(3)} \begin{pmatrix} 1 & 0 & 0 & \frac12 & 4 & -\frac32 \\ 0 & 1 & 0 & \frac12 & 1 & -\frac12 \\ 0 & 0 & 1 & 0 & -2 & 1 \end{pmatrix}.$$

Therefore,

$$M^{-1} = \begin{pmatrix} \frac12 & 4 & -\frac32 \\ \frac12 & 1 & -\frac12 \\ 0 & -2 & 1 \end{pmatrix}.$$

Check by calculating either MM^{-1} or $M^{-1}M$ and getting the 3×3 identity matrix.