Math. 311 (Fulling) 8 May 2000

Final Examination – Solutions (corrected)

1. *(25 pts.)*

(a) Find all the eigenvalues and eigenvectors of $B = \begin{pmatrix} 1 & 5 \\ 3 & 3 \end{pmatrix}$.

$$0 = \begin{vmatrix} 1-\lambda & 5 \\ 3 & 3-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3) - 15 = \lambda^2 - 4\lambda - 12 = (\lambda-6)(\lambda+2),$$

so the eigenvalues are 6 and -2.

$\underline{\lambda = 6}$: $B - \lambda = \begin{pmatrix} -5 & 5 \\ 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x - y = 0.$

Thus the eigenvectors are the multiples of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\underline{\lambda = -2}$: $B - \lambda = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{5}{3} \\ 0 & 0 \end{pmatrix} \Rightarrow x + \frac{5}{3} y = 0.$

Thus the eigenvectors are the multiples of $\vec{v}_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

(b) Solve the differential equation system

$$\frac{dx}{dt} = x + 5y, \qquad x(0) = -1,$$
$$\frac{dy}{dt} = 3x + 3y, \qquad y(0) = 3.$$

Method 1: $\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -5 \\ 3 \end{pmatrix} e^{-2t}$.

To find the coefficients, apply the initial conditions:

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 - 5c_2 \\ c_1 + 3c_2 \end{pmatrix}.$$

Solve for c_1 and c_2:

$$\begin{pmatrix} 1 & -5 & -1 \\ 1 & 3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -5 & -1 \\ 0 & 8 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Thus $c_1 = \frac{3}{2}$, $c_2 = \frac{1}{2}$.

Method 2: Let $U = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix}$, the matrix whose columns are the eigenvectors. Then $U^{-1} = \frac{1}{8} \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}$. The solution is

$$e^{tB} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = U e^{tD} U^{-1} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \frac{1}{8} \begin{pmatrix} 12 \\ 4 \end{pmatrix}$$
$$= \begin{pmatrix} e^{6t} & -5e^{-2t} \\ e^{6t} & 3e^{-2t} \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix},$$

which is equivalent to the result of the other method.

2. *(10 pts.)* Explain the following statement: The identity $\nabla\cdot(\nabla\times\vec{B})=0$ is necessary to prevent an inconsistency between Stokes's theorem and Gauss's theorem. HINT: Consider two surfaces that have the same curve as boundary.

Stokes implies that $\iint(\nabla\times\vec{B})\cdot d\vec{S}=\oint\vec{B}\cdot d\vec{r}$, which is **the same** for both surfaces. Gauss implies that the difference between the two surface integrals is $\iiint\nabla\cdot(\nabla\times\vec{B})\,dx\,dy\,dz$ over the volume between the surfaces. If $\nabla\cdot(\nabla\times\vec{B})\neq 0$, this can't be 0 for all possible pairs of surfaces, which is a contradiction.

3. *(30 pts.)* The linear function $L\colon\mathbf{R}^3\to\mathbf{R}^2$ is represented by the matrix $N=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$ (with respect to the natural basis).

(a) Is L injective (1-to-1)? If not, what is its kernel?

NO. The kernel is the solutions of the homogeneous equation (always nontrivial when there are more unknowns than equations), which we find by reducing the matrix:

$$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\to\begin{pmatrix}1&2&3\\0&-4&-8\end{pmatrix}\to\begin{pmatrix}1&0&-1\\0&1&2\end{pmatrix}\Rightarrow\begin{array}{c}x-z=0,\\y+2z=0.\end{array}$$

Thus the kernel consists of multiples of $\begin{pmatrix}1\\-2\\1\end{pmatrix}$.

(b) Is L surjective (onto)? If not, what is its range?

YES. (The range is the span of the columns, which is all of $\mathbf{R}^2$ in this case.)

(c) What matrix represents L with respect to the bases

$$\left\{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}\ \text{for the domain and}\ \left\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}\ \text{for the codomain?}$$

Let $H=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$. Since $H=H^{\mathrm{t}}=H^{-1}=H^{-1\mathrm{t}}$, we need not agonize over which of these 4 matrices to use to transform the domain. For the codomain, let $G=\begin{pmatrix}1&0\\2&1\end{pmatrix}$; it maps the new coordinates into the natural coordinates, so what we really need is $G^{-1}=\begin{pmatrix}1&0\\-2&1\end{pmatrix}$. Finally, the desired matrix is

$$G^{-1}NH=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0\\-2&1\end{pmatrix}\begin{pmatrix}3&2&1\\1&2&3\end{pmatrix}=\begin{pmatrix}3&2&1\\-5&-2&1\end{pmatrix}.$$

4. *(15 pts.)* Show that if $\vec{B}$ is a constant vector field (that is, $\vec{B}(\vec{r})$ is independent of $\vec{r}$, where $\vec{r} \equiv x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ as usual), then $\vec{A}(\vec{r}) \equiv \frac{1}{2}\vec{B} \times \vec{r}$ is a vector potential for $\vec{B}$:

$$\nabla \times \vec{A} = \vec{B}.$$

METHOD 1: Use the identity for simplifying the triple cross product, keeping in mind that ∇ operates on $\vec{r}$ in both terms:

$$\nabla \times (\vec{B} \times \vec{r}) = (\nabla \cdot \vec{r})\vec{B} - (\vec{B} \cdot \nabla)\vec{r}.$$

Now $\nabla\vec{r}$ (the gradient or Jacobian matrix of $\vec{r}$ with respect to itself) is just the identity matrix, so $\vec{B} \cdot \nabla\vec{r}$ is just $\vec{B}$, and the divergence $\nabla \cdot \vec{r}$ equals 3. Thus

$$\nabla \times (\vec{B} \times \vec{r}) = 3\vec{B} - \vec{B} = 2\vec{B},$$

as asserted.

METHOD 2: Let's write it out.

$$2\vec{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \hat{\imath}(B_y z - B_z y) + \hat{\jmath}(B_z x - B_x z) + \hat{k}(B_x y - B_y x).$$

Thus

$$\nabla \times (2\vec{A}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \\ (B_y z - B_z y) & (B_z x - B_x z) & (B_x y - B_y x) \end{vmatrix}$$
$$= \hat{\imath}(B_x + B_x) + \hat{\jmath}(B_y + B_y) + \hat{k}(B_z + B_z) = 2\vec{B}.$$

5. *(25 pts.)* Find an **orthonormal** basis of eigenvectors for $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -2 \end{pmatrix}$.

HINT: One of the eigenvalues should be obvious.

The obvious eigenvalue is 2; we shall be careful to keep the factor $(2 - \lambda)$ in the determinant separate.

$$0 = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 2 & -2-\lambda \end{vmatrix} = (2-\lambda)[((\lambda-1)(\lambda+2)-4] = -(\lambda-2)(\lambda^2+\lambda-6) = -(\lambda-2)^2(\lambda+3).$$

The eigenvalues are 2 and -3.

$$\underline{\lambda = 2}: \quad A - \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -4 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad \begin{array}{l} x \text{ arbitrary}, \\ y - 2z = 0. \end{array}$$

We can pick one vector in the x direction and another with no x component (so that they will be orthogonal) and normalize them:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$

$$\underline{\lambda = -3}: \quad A - \lambda = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \qquad \begin{aligned} x &= 0, \\ y + \tfrac{1}{2}z &= 0. \end{aligned}$$

Thus a normalized eigenvector is

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}.$$

Of course, it is automatically orthogonal to the other basis elements, because the matrix is symmetric. It is interesting to note that we could have found this vector as the cross product of the two eigenvectors for $\lambda = 2$, without even bothering to solve the characteristic equation. Then we could apply the matrix to it and observe that the eigenvalue is -3 .

6. *(15 pts.)* A car contains 1000 cubic inches of steel and 30 cubic inches of aluminum. A bicycle contains 25 cubic inches of steel and 2 cubic inches of aluminum. Steel weighs 2 pounds per cubic inch and costs \$3 per cubic inch. Aluminum weighs 1 pound per cubic inch and costs \$5 per cubic inch. Organize these facts into matrices, and find the matrix that should be used to calculate the total weight and total cost of the material needed to make x cars and y bicycles.

Let s , a , w , and c have their obvious meanings. The first two sentences tell how much metal is needed, and the next two tell how to find the weight and cost of given amounts of metal:

$$\begin{pmatrix} s \\ a \end{pmatrix} = \begin{pmatrix} 1000 & 25 \\ 30 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \qquad \begin{pmatrix} w \\ c \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} s \\ a \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w \\ c \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 1000 & 25 \\ 30 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2030 & 52 \\ 3150 & 85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

7. *(20 pts.)* Consider the two nonlinear equations in four variables,

$$vx^2 + u^2y = 2, \qquad uy^2 + v^2 = 0.$$

They implicitly define $\binom{u}{v}$ as a function of $\binom{x}{y}$. Find the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, at the point $x = 1$, $y = 1$, $u = -1$, $v = 1$.

Differentiate both equations with respect to x :

$$\begin{aligned} x^2 \frac{\partial v}{\partial x} + 2xv + 2yu \frac{\partial u}{\partial x} &= 0, \\ y^2 \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} &= 0. \end{aligned}$$

At this point it is convenient to substitute the numerical values of the variables, and then rearrange:

$$\begin{aligned} -2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} &= -2, \\ \frac{\partial u}{\partial x} + 2 \frac{dv}{dx} &= 0. \end{aligned}$$

Since we know that the same coefficient matrix will appear in the $\frac{\partial}{\partial y}$ equations, let's solve this system by calculating the inverse matrix:

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{-5}\begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \equiv G^{-1}.$$

Then

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix} = G^{-1}\begin{pmatrix} -2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ -\frac{2}{5} \end{pmatrix}.$$

Now differentiate the original equations with respect to y :

$$x^2\,\frac{\partial v}{\partial y} + u^2 + 2uy\,\frac{\partial u}{\partial y} = 0,$$
$$2uy + y^2\,\frac{\partial u}{\partial y} + 2v\,\frac{\partial v}{\partial y} = 0;$$

$$-2\,\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -1,$$
$$\frac{\partial u}{\partial y} + 2\,\frac{\partial v}{\partial y} = 2;$$

$$\begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \end{pmatrix} = G^{-1}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \end{pmatrix}.$$

8. *(20 pts.)* Do **ONE** of these [(A) or (B)]; 10 points extra credit for doing the other one.

(A) The *index* of a matrix M is defined as

$$\operatorname{index}(M) \equiv \dim(\ker M) - \dim(\ker M^{\mathrm{t}}).$$

Show that the index of a $p \times n$ matrix is always equal to $n - p$. Hint: Use the fact that

row rank = column rank

and another famous theorem of a similar nature.

The other theorem is

$$\dim(\ker M) + \dim(\operatorname{ran} M) = \dim(\operatorname{dom} M),$$

which applies equally well to M^{t} . Thus

$$\dim(\ker M) = \dim(\operatorname{dom} M) - \dim(\operatorname{ran} M) = n - \text{column rank}$$

and

$$\dim(\ker M^{\mathrm{t}}) = \dim(\operatorname{dom} M^{\mathrm{t}}) - \dim(\operatorname{ran} M^{\mathrm{t}}) = p - \text{row rank}.$$

Subtract:

$$\dim(\ker M) - \dim(\ker M^{\mathrm{t}}) = n - p.$$

(B) Calculate this determinant, **showing all steps**.

Calculators may be used for elementary arithmetic only!

$$\begin{vmatrix} 1 & 3 & 5 & 1 & -1 \\ 1 & 6 & 1 & -1 & 2 \\ 0 & 3 & 0 & 0 & 0 \\ 2 & 6 & 0 & 3 & 1 \\ 4 & 9 & 8 & 0 & 4 \end{vmatrix}$$

Expand in cofactors of the middle row:

$$-3\begin{vmatrix} 1 & 5 & 1 & -1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 3 & 1 \\ 4 & 8 & 0 & 4 \end{vmatrix}.$$

Extract a factor 4 from the bottom row, and move that row to the top (3 transpositions), since it is the simplest row to pivot from:

$$+12\begin{vmatrix} 1 & 2 & 0 & 1 \\ 1 & 5 & 1 & -1 \\ 1 & 1 & -1 & 2 \\ 2 & 0 & 3 & 1 \end{vmatrix}.$$

Use the first row to clear out the first column:

$$12\begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & -1 & -1 & 1 \\ 0 & -4 & 3 & -1 \end{vmatrix} = 12\begin{vmatrix} 3 & 1 & -2 \\ -1 & -1 & 1 \\ -4 & 3 & -1 \end{vmatrix}.$$

Move the middle row to the top, extract a compensating sign, and use it to clear out the new first column:

$$+12\begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & -2 \\ -4 & 3 & -1 \end{vmatrix} = 12\begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 1 \\ 0 & 7 & -5 \end{vmatrix}.$$

The rest is elementary:

$$12\begin{vmatrix} -2 & 1 \\ 7 & -5 \end{vmatrix} = 12(10-7) = 36.$$

Of course, many other sequences of operations are equally correct.

9. *(25 pts.)* Do **ONE** of these [(A) or (B)]; 15 points extra credit for doing the other one.

(A) Let

$$\vec{F}(\vec{r}) = \frac{x}{r^3}\,\hat{\imath} + \frac{y}{r^3}\,\hat{\jmath} + e^{x^3+y^3}\,\hat{k}.$$

Calculate the surface integral $\iint_S \vec{F}\cdot d\vec{S}$ through the cylindrical surface S :

$$0 < z < 3, \quad r \equiv \sqrt{x^2+y^2} = 2, \quad 0 \le \theta < 2\pi.$$

(The end faces of the "can" are **not** included in S . (r,θ) are polar coordinates in the (x,y) plane.) HINT: What is the unit normal vector to the surface?

METHOD 1: The unit normal is the unit vector in the direction of $x\hat\imath + y\hat\jmath$:

$$\hat n = \frac{x\hat\imath + y\hat\jmath}{r}, \quad \text{hence} \quad \vec F\cdot\hat n = \frac{x^2+y^2}{r^4} = \frac{1}{r^2} = \frac14 .$$

So the integral is just $\frac14$ times the area of the surface:

$$\iint_S \vec F\cdot d\vec S = \int_0^3 dz \int_0^{2\pi} r\,d\theta\,\frac14 = 3\cdot 2\cdot(2\pi)\cdot\frac14 = 3\pi.$$

METHOD 2: Use $x = r\cos\theta$, $y = r\sin\theta$, where r is constant ($=2$), so

$$dx = -2\sin\theta\,d\theta, \qquad dy = 2\cos\theta\,d\theta.$$

Now we can convert

$$\iint_S \vec F\cdot d\vec S = \iint [F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy]$$

to an integral over θ and z , using $(d\theta)^2 = 0$ and $dz\,d\theta = -d\theta\,dz$.

$$\begin{aligned}\iint_S \vec F\cdot d\vec S &= \iint_S \left[\frac{x}{r^3}\,2\,\cos\theta + \frac{y}{r^3}\,2\,\sin\theta + 0\right] d\theta\,dz \\ &= \int_0^3 dz \int_0^{2\pi} d\theta\,\frac{2\cdot 2}{8}\,(\cos^2\theta + \sin^2\theta) = 3\pi.\end{aligned}$$

COMMENT: Several students pointed out that because F_z does not depend on z , the fluxes through the top and bottom endfaces of the “can” cancel each other, and therefore Gauss’s theorem can be used to express the integral as the volume integral of $\nabla\cdot\vec F$ over the interior of the can, after all. Unfortunately, the divergence of $\vec F$ is rather messy to calculate in this problem (unless you know the formula for doing that in cylindrical coordinates). The divergence is **not** $2/r^3$, and even if it were, it would not be correct to substitute $r = 2$, because we would now need to integrate over the interior, not just the surface. (This pair of errors was seen on several papers.) Finally, this vector field has a singularity at $r = 0$ (somewhat like a line of charges along the axis of the cylinder), and just as for a Coulomb field this will spoil Gauss’s theorem unless taken into account in some special way.

(B) Consider the inner product

$$\langle p, q\rangle \equiv \int_{-\infty}^{\infty} p(t)\,q(t)\,e^{-t^2}\,dt$$

on the vector space of all polynomials. The orthonormal polynomials with respect to this inner product are called *Hermite polynomials*. Find the first 3 Hermite polynomials. (In other words, apply the Gram–Schmidt algorithm to the powers $\{1, t, t^2, \cdots\}$, stopping after t^2 .) FREE INFORMATION:

$$\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt\pi, \quad \int_{-\infty}^{\infty} t^2\,e^{-t^2}\,dt = \frac{\sqrt\pi}{2}, \quad \int_{-\infty}^{\infty} t^4\,e^{-t^2}\,dt = \frac{3\sqrt\pi}{4}.$$

Note first that whenever m is *odd*,

$$\int_{-\infty}^{\infty} t^m e^{-t^2}\, dt = 0$$

by symmetry (the contribution from negative t exactly cancels that from positive t), which is why those integrals did not need to be tabulated in the hint. The square of the norm of the first vector is

$$\langle 1, 1\rangle = \int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi},$$

so

$$\hat{u}_0 = \pi^{-1/4}.$$

Next,

$$\langle 1, t\rangle = \int_{-\infty}^{\infty} t\, e^{-t^2}\, dt = 0,$$

so the second vector is already orthogonal to the first; all we need to do is normalize it:

$$\langle t, t\rangle = \int_{-\infty}^{\infty} t^2 e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\,,$$

so

$$\hat{u}_1 = \sqrt{2}\,\pi^{-1/4}\, t.$$

At the third step we have $\langle \hat{u}_1, t^2\rangle = 0$ but

$$\langle \hat{u}_0, t^2\rangle = \pi^{-1/4}\cdot \frac{\sqrt{\pi}}{2}\,,$$

so the part of t^2 parallel to $\hat{u}_0$ is $\frac{1}{2}$. Thus the perpendicular part is

$$t^2_\perp = t^2 - \tfrac{1}{2}\,,$$

so that

$$\begin{aligned}\langle t^2_\perp, t^2_\perp\rangle &= \int_{-\infty}^{\infty} (t^4 - t^2 + \tfrac{1}{4})\, e^{-t^2}\, dt \\ &= \frac{3\sqrt{\pi}}{4} - \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{4} = \frac{\sqrt{\pi}}{2}\,.\end{aligned}$$

Therefore,

$$\hat{u}_2 = \sqrt{2}\,\pi^{-1/4}\left(t^2 - \frac{1}{2}\right).$$

10. **Take 15 free points in honor of your classmates who will graduate this week!**