Math. 311 (Fulling) 16 February 2000

Test A – Solutions (corrected)

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* Find all solutions of the linear system $\begin{cases} x + 2y + 3z = 2, \\ 3x + 6y - \;\; z = -4, \\ -2x - 4y + \;\; z = 3. \end{cases}$

Form the augmented matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 3 & 6 & -1 & -4 \\ -2 & -4 & 1 & 3 \end{pmatrix}.$$

Subtract appropriate multiples of the first row from the other two:

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & -10 & -10 \\ 0 & 0 & 7 & 7 \end{pmatrix}.$$

Use the second row to cancel the third one, divide the second row by -10 , and subtract 3 times it from the first row:

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding system of equations is

$$x + 2y = -1, \qquad z = 1.$$

The general solution is

$$x = -1 - 2y, \quad z = 1, \quad \text{where } y \text{ is an arbitrary parameter.}$$

2. *(10 pts.)* Express $t^2 - 3t + 16$ as a linear combination of the powers of the quantity $(t+2)$.

Let $s = t + 2$, so $t = s - 2$. Then the given polynomial is

$$\begin{aligned}(s-2)^2 - 3(s-2) + 16 &= s^2 - 4s + 4 - 3s + 6 + 16 \\ &= s^2 - 7s + 26.\end{aligned}$$

Substitute back in, getting

$$(t+2)^2 - 7(t+2) + 26.$$

3. *(15 pts.)* The prices of zarfs and bibcocks are being forced up! Steel now costs \$3 per kilo, while aluminum is \$5 per kilo.

   What SHOULD have been the next sentence (Version A): In addition, there is a sales tax of \$1 on each kilo of metal of any kind.

   What DID appear on the test (Version B): In addition, there is a sales tax of \$1 on each zarf and each bibcock.

   A zarf requires 1 kilogram each of steel and aluminum. A bibcock takes 2 kilos of steel and 3 of aluminum.

   Organize these facts into matrices, and **find the matrix** that should be used to calculate the the total raw material cost and the total sales tax for z zarfs and b bibcocks. (Keep the cost and the tax separate, but add the zarf and bibcock contributions to each.)

Solution to Version A *(a routine problem):* With the obvious notations for the six scalar quantities, let's define matrices A and B this way:

$$\begin{pmatrix} c \\ t \end{pmatrix} = A \begin{pmatrix} s \\ a \end{pmatrix}, \qquad \begin{pmatrix} s \\ a \end{pmatrix} = B \begin{pmatrix} z \\ b \end{pmatrix}.$$

Reading the problem statement carefully, we see that

$$A = \begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} c \\ t \end{pmatrix} = AB \begin{pmatrix} z \\ b \end{pmatrix}, \qquad \text{where } AB = \begin{pmatrix} 3 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 8 & 21 \\ 2 & 5 \end{pmatrix}.$$

Solution to Version B *(a slightly weird problem):* Define the matrix B as above, and again conclude that

$$B = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

Reading the problem statement still more carefully, we see that the tax doesn't fit naturally into matrix A. Instead, we can define a row matrix $A_1 = (\,3 \quad 5\,)$ so that $c = A_1 \begin{pmatrix} s \\ a \end{pmatrix}$. Therefore,

$$c = A_1 B \begin{pmatrix} z \\ b \end{pmatrix}, \qquad \text{where } A_1 B = (\,3 \quad 5\,) \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = (\,8 \quad 21\,),$$

or, in other words, $c = 8z + 21b$. The tax must be computed directly from z and b as

$$t = (\,1 \quad 1\,) \begin{pmatrix} z \\ b \end{pmatrix} = z + b.$$

We can put these two results together in an answer of the requested form,

$$\begin{pmatrix} c \\ t \end{pmatrix} = C \begin{pmatrix} z \\ b \end{pmatrix}, \qquad \text{where } C = \begin{pmatrix} 8 & 21 \\ 1 & 1 \end{pmatrix}.$$

ALTERNATIVE SOLUTION TO VERSION B *(found on 2 student papers):* Compute the tax along with the materials:

$$\begin{pmatrix} s \\ a \\ t \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} z & b \end{pmatrix}.$$

Then just carry the tax through the next step:

$$\begin{pmatrix} c \\ t \end{pmatrix} = \begin{pmatrix} 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ a \\ t \end{pmatrix}.$$

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$$C = \begin{pmatrix} 3 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 21 \\ 1 & 1 \end{pmatrix}.$$

4. *(13 pts.)* Let's define a mapping T of the function space $\mathcal{C}^2(0,1)$ into the function space $\mathcal{C}(0,\infty)$ by

$$T(f)(t) \equiv \frac{d^2 f}{dt^2} + t^2 f(t).$$

(Here t is the independent variable of the functions in $\mathcal{C}^1(0,1)$, and f is a generic element of $\mathcal{C}^1(0,1)$.) Is T linear, affine, or nonlinear? Justify your answer.

Linear, because

$$\begin{aligned} T(cf_1 + f_2) &= \frac{d^2}{dt^2}(cf_1 + f_2) + t^2(cf_1 + f_2) \\ &= cf_1'' + f_2'' + ct^2 f_1 + t^2 f_2 = cT(f_1) + T(f_2). \end{aligned}$$

5. *(12 pts.)* Calculate $\frac{d}{dx} \int_1^{x^2} \frac{\cos(tx)}{t}\, dt$.

By the multivariable chain rule, there are two terms, one from the x dependence of the upper limit and one from the x dependence of the integrand. The first term, by the fundamental theorem of calculus, is

$$(2x)\frac{\cos(x^3)}{x^2} = \frac{2\cos(x^3)}{x}.$$

The second term is

$$\int_1^{x^2} \frac{-t\sin(tx)}{t}\, dt = -\int_1^{x^2} \sin(tx)\, dt = \frac{\cos(tx)}{x}\bigg|_{t=1}^{x^2} = \frac{\cos(x^3)}{x} - \frac{\cos x}{x}.$$

The total is

$$\frac{3\cos(x^3) - \cos x}{x}.$$

6. *(15 pts.)* The temperature in a metal plate with coordinates (x, y) is

$$T = 300 + 3x^2 + x \sin y.$$

(a) Find a vector that points in the direction of fastest increase of T at the point $(x, y) = (1, 0)$.

Such a vector is the gradient,

$$\nabla T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right) = (6x + \sin y, x \cos y),$$

evaluated at $(1, 0)$:

$$\nabla T(1, 0) = (6, 1).$$

(b) Construct the best affine approximation (also known as the first-order approximation) to T in the neighborhood of the point $(x_0, y_0) = (1, 0)$.

First note that $T(1, 0) = 300 + 3 + 0 = 303$. Thus

$$T + \nabla T \approx T(x_0, y_0) + \nabla T(x_0, y_0) \cdot (x - x_0, y - y_0) = 303 + 6(x - 1) + y$$

by use of the answer to (a).

7. *(20 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 3 & 8 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{pmatrix}$.

The augmented matrix is

$$\begin{pmatrix} 3 & 8 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{pmatrix}.$$

We minimize arithmetic grief by putting the middle row at the top. (Note that it is essential to do this to the identity matrix on the right as well as the original matrix on the left!)

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 8 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 0 & 1 \end{pmatrix}.$$

Subtract the obvious multiples of the first row from the others;

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 8 & -1 & 1 & -3 & 0 \\ 0 & -1 & 1 & 0 & -2 & 1 \end{pmatrix}.$$

Here the second row is the negative of the old third row, and the third row is the old second row plus 8 times the old third row:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 7 & 1 & -19 & 8 \end{pmatrix}.$$

Now divide the third row by 7 and add or subtract from the top two rows to get

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{1}{7} & \frac{26}{7} & -\frac{8}{7} \\ 0 & 1 & 0 & \frac{1}{7} & -\frac{5}{7} & \frac{1}{7} \\ 0 & 0 & 1 & \frac{1}{7} & -\frac{19}{7} & \frac{8}{7} \end{pmatrix}.$$

Thus

$$M^{-1} = \frac{1}{7}\begin{pmatrix} -1 & 26 & -8 \\ 1 & -5 & 1 \\ 1 & -19 & 8 \end{pmatrix}.$$