Math. 311 (Fulling) 24 March 2000

Test B – Solutions

1. *(12 pts.)* Tell whether each of these sets is linearly independent. (Justify your answers briefly.)

   (a) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} \pi \\ -\pi \end{pmatrix} \right\}$

NO — 3 independent vectors can't fit into a 2-dimensional space.

   (b) $\{\sinh t, \quad \cosh t, \quad e^{2t}, \quad e^{-2t}\}$

YES — $\sinh t$ and $\cosh t$ can be written *linearly* in terms of $e^{\pm t}$, but *not* in terms of $e^{\pm 2t}$.

2. *(18 pts.)* The matrix $M = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 10 \end{pmatrix}$ row-reduces to $M_{\text{red}} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 4 \end{pmatrix}$. (This is free information!)

   (a) What is the dimension of the kernel of M ?

1 , because x_3 is a free parameter but x_1 and x_2 are then determined.

   (b) What is the rank of M ?

2 . This is $3 - 1$ (the dimension of the domain minus the dimension of the kernel). Alternatively, just notice that the span of the columns is obviously all of $\mathbf{R}^2$.

   (c) Suppose that you could change the numbers in M (but leave it a 2×3 matrix). Could you make the rank larger? Could you make the rank smaller? (Explain.)

The rank can't be larger that 2 , since the range is a subspace of $\mathbf{R}^2$. The rank could be 1 or 0 — if, for instance,

$$M_{\text{red}} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3. *(20 pts.)* Let $\mathcal{V}$ be the span of the functions $\{\vec{b}_1 = 1, \quad \vec{b}_2 = \cos t, \quad \vec{b}_3 = \cos(2t)\}$. Define $L\colon \mathcal{V} \to \mathcal{V}$ by

$$L(f) = f'' + 4f.$$

   (a) Find the matrix representing L with respect to the basis $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$.

Calculate the effect of L on the basis elements:

$$L(1) = 4, \quad L(\cos t) = -\cos t + 4\cos t = 3\cos t, \quad L(\cos 2t) = -4\cos 2t + 4\cos 2t = 0.$$

By the k th-column rule, therefore, the matrix is

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) Is L injective? If not, what is its kernel?

NO. The kernel consists of the multiples of $\cos 2t$. (This can be seen immediately from the matrix above.)

(c) Is L surjective? If not, what is its range?

NO. The range is the span of the basis elements t and $\cos t$ — that is, the functions of the form $\{a + b\cos t\}$ for scalars a and b.

4. *(15 pts.)* Do **ONE** of these [(A) or (B)]. (Up to 5 points extra credit for doing both.)

(A) Because $\cos(2t) = 2\cos^2 t - 1$, an alternative basis for the vector space $\mathcal{V}$ in the previous problem is $\{\vec{c}_1 = 1, \quad \vec{c}_2 = \cos t, \quad \vec{c}_3 = \cos^2 t\}$. Find the matrix that maps **coordinates** of vectors with respect to the b-basis into coordinates with respect to the c-basis.

We have

$$\vec{b}_1 = \vec{c}_1\,,$$
$$\vec{b}_2 = \vec{c}_2\,,$$
$$\vec{b}_3 = -\vec{c}_1 + 2\vec{c}_3\,.$$

That is, the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

maps the $\vec{c}$ vectors to the $\vec{b}$ vectors. Therefore, its transpose,

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

maps the b-coordinates to the c-coordinates, which is what we want.

(B) Find a basis for the span of

$$\{(0,3,0,3), \quad (2,3,4,5), \quad (5,5,3,3), \quad (-1,-1,-1,-1)\}.$$

Let us make a matrix out of these row vectors and row-reduce. At the first step I rotated the bottom row to the top and used it to zero out the rest of the first column. The rest of the steps are analogous and fairly obvious.

$$\begin{pmatrix} 0 & 3 & 0 & 3 \\ 2 & 3 & 4 & 5 \\ 5 & 5 & 3 & 3 \\ -1 & -1 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows (now obviously independent) are a basis for the span:

$$\{(1,0,1,0), \quad (0,1,0,1), \quad (0,0,1,1)\}.$$

There are other correct answers; for example, if you reduce all the way to Gauss–Jordan form, you will get

$$\{(1,0,0,-1), \quad (0,1,0,1), \quad (0,0,1,1)\}.$$

5. *(20 pts.)* The linear function $F\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ with respect to the natural basis.

(a) Find the matrix representing F if the basis in the codomain is changed to

$$\mathcal{B} = \left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

(the basis for the domain remaining unchanged).

The matrix $B \equiv= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ maps the $\mathcal{B}$-coordinates to the natural coordinates. Therefore, $B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ maps natural coordinates to $\mathcal{B}$-coordinates. Therefore the desired matrix is

$$B^{-1}A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix}.$$

(b) Find the matrix representing F if the basis in the domain is changed to $\mathcal{B}$ (the basis for the codomain remaining unchanged).

Use the matrix B from above. The desired matrix is

$$AB = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 4 & 2 \end{pmatrix}.$$

(c) Find the matrix representing F if the basis $\mathcal{B}$ is used for both domain and codomain.

The correct matrix is $B^{-1}AB$. It can be calculated using the result of either (a) or (b) as an intermediate step. I'll do the former:

$$(B^{-1}A)B = \frac{1}{2}\begin{pmatrix} 3 & 3 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}.$$

6. *(15 pts.)* Do **ONE** of the following [(A) or (B)]. (Up to 10 points extra credit for doing both.)

(A) Prove that the *range* of a linear function is a *subspace* (of the codomain).

This is in the textbook: Theorem 1 of Sec. 5.2.

(B) Consider the linear problem consisting of the differential equation and boundary conditions

$$\frac{d^2y}{dt^2} + 4y = f(t), \qquad y(0) = 0, \quad y(\pi) = 0.$$

Here $f(t)$ is some given function, such as $\sin t$, and the variable t runs from 0 to π. **Without actually solving the problem**, discourse learnedly upon the general properties of its set of solutions, flinging around terms like *domain*, *range*, *kernel*, *affine subspace*, *homogeneous*, *superposition*, *injective*, and *surjective*. HINT: Choose the domain of the linear operator to be the subspace of $\mathcal{C}^2(0,\pi)$ consisting of functions satisfying the boundary conditions, $y(0) = 0$ and $y(\pi) = 0$.

The problem consists of a nonhomogeneous ordinary differential equation together with two homogeneous boundary conditions. Let $\mathcal{D}$ be the domain defined in the hint. The range is then the set of functions f for which the problem can be solved, the solution y being in $\mathcal{D}$. For a given f, the space of solutions of this nonhomogeneous problem is an affine subspace of $\mathcal{D}$. It can be constructed by superposition of any particular solution of the nonhomogeneous problem with all the solutions of the corresponding homogeneous problem, $y'' + 4y = 0$ with the same boundary conditions. This space of homogeneous solutions is, by definition, the kernel of the linear operator. The operator will be injective if this kernel consists only of the zero function. It will be surjective if the problem can be solved for *all* functions f in some reasonable space (say $\mathcal{C}(0,\pi)$).

You could go beyond the call of duty by observing that the kernel consists precisely of the multiples of the function $\sin 2t$, so the operator is *not* injective. (The kernel is not the usual two-dimensional space of all homogeneous solutions, $c_1 \cos 2t + c_2 \sin 2t$, because in this problem the boundary conditions force $c_1 = 0$.) Furthermore — although there is no reason why you should be expected to know this — a solution of the nonhomogeneous problem exists only if

$$\int_0^\pi f(t) \sin 2t \, dt = 0.$$

(This condition says that f must be *orthogonal* to the vectors in the kernel, with respect to a certain, very natural, inner product. This is not an accident; compare Theorem 5 in Sec. 8.2.)