Math. 311 (Fulling) 19 April 2000

Test C – Solutions (corrected)

Calculators may be used for simple arithmetic operations only!

1. *(30 pts.)* Let $\vec{F} = (x-y)\hat{\imath} + z\hat{\jmath} + (z-y)\vec{k}$.

(a) Calculate $\nabla \cdot \vec{F}$.

$$\frac{\partial(x-y)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial(z-y)}{\partial z} = 1 + 0 + 1 = 2.$$

(b) Calculate $\nabla \times \vec{F}$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ (x-y) & z & (z-y) \end{vmatrix} = \hat{\imath}(-1-1) + \hat{\jmath}(0-0) + \hat{k}(0+1) = -2\,\hat{\imath} + \hat{k}.$$

(c) Calculate $\iint_R \vec{F}\cdot d\vec{S}$ when R is the portion of the plane $z = 2-5x$ that lies in the quadrant $z>0$, $x>0$ and between the planes $y=0$ and $y=1$.

Note first that the equation of the surface can also be written as

$$x = \frac{2-z}{5}\,,$$

and that when $x=0$, $z=2$, and when $z=0$, $x=\frac{2}{5}$. Thus the integration will be from 0 to 2 in z *or* from 0 to $\frac{2}{5}$ in x . Now let's write the integral as

$$\iint [F_x\,dy\,dz + F_y\,dz\,dx + F_z\,dx\,dy]$$

and ponder how best to integrate each term.

Method 1: Integrate each term over its own plane. The projection onto the $x-z$ plane has zero area, so the F_y term is zero. The others are

$$\int_0^1 dy \int_0^2 dz\,(x-y) + \int_0^1 dy \int_0^{2/5} dx\,(z-y) = \int_0^1 dy \int_0^2 dz\left(\frac{2-z}{5} - y\right) + \int_0^1 dy \int_0^{2/5} dx\,(2-5x-y)$$

$$= \int_0^2 dz\left[\frac{2-z}{5}\,y - \frac{y^2}{2}\right]_0^1 + \int_0^1 dy\left[2x - \frac{5x^2}{2} - yx\right]_0^{\frac{2}{5}} = \int_0^2 dz\left[\frac{2-z}{5} - \frac{1}{2}\right] + \int_0^1 dy\left[\frac{4}{5} - \frac{2}{5} - \frac{2y}{5}\right]$$

$$= \left[-\frac{z}{10} - \frac{z^2}{10}\right]_0^2 + \left[\frac{2y}{5} - \frac{y^2}{5}\right]_0^1 = -\frac{1}{5} - \frac{2}{5} + \frac{2}{5} - \frac{1}{5} = -\frac{2}{5}\,.$$

Method 2: Integrate everything over x and y . We have $dz = -5\,dx + 0\,dy$. Therefore, the integral is

$$\iint_D \left[(x-y)\,dy\,(-5\,dx) + F_y\,(dx)^2 + (z-y)\,dx\,dy\right] = \int_0^1 dy \int_0^{2/5} dx\,[+5(x-y) + (2-5x-y)]$$

$$= \int_0^1 dy \int_0^{2/5} dx\,[5x - 5y + 2 - 5x - y] = \int_0^1 dy \int_0^{2/5} dx\,[-6y+2]$$

$$= \int_0^{2/5} dx\,\left[-3y^2 + 2y\right]_0^1 = \frac{2}{5}(-3+2) = -\frac{2}{5}\,.$$

(d) Calculate $\iint_S \vec{F} \cdot d\vec{S}$ when S is the sphere $(x-1)^2 + y^2 + (z+2)^2 = 25$.

By Gauss's theorem, this is the integral of $\nabla\cdot\vec{F}$ over the ball whose boundary is S . Since $\nabla\cdot\vec{F} = 2$, this is just twice the volume of the ball. Since the radius is 5 , this is

$$2 \cdot \frac{4\pi}{3} \cdot 5^3 = \frac{8\pi}{3} \cdot 125 = \frac{1000\pi}{3}.$$

2. *(10 pts.)* Find the volume of the parallelepiped generated by the edges

$$\vec{v}_1 = (1,2,1), \quad \vec{v}_2 = (2,0,2), \quad \vec{v}_3 = (1,2,3).$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -2 \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = -2(6-2) - 0 = -8.$$

So the volume is $+8$.

3. *(30 pts.)* Define curvilinear coordinates (u,t) by $\begin{cases} x = e^u \cosh t, \\ y = e^u \sinh t. \end{cases}$

(a) Find the formulas for the tangent vectors to the coordinate curves (at a generic point (u,t)).

$$\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} e^u \cosh t \\ e^u \sinh t \end{pmatrix}, \quad \frac{\partial \vec{r}}{\partial t} = \begin{pmatrix} e^u \sinh t \\ e^u \cosh t \end{pmatrix}.$$

For future use, let us put these together (as *columns*) into the Jacobian matrix,

$$J = \begin{pmatrix} e^u \cosh t & e^u \sinh t \\ e^u \sinh t & e^u \cosh t \end{pmatrix}.$$

(b) Find the formulas for the normal vectors to the coordinate "surfaces" (which are actually curves in this two-dimensional case).

These are the *rows* of J^{-1} . So we start with

$$\det J = \begin{vmatrix} e^u \cosh t & e^u \sinh t \\ e^u \sinh t & e^u \cosh t \end{vmatrix} = e^{2u}(\cosh^2 t - \sinh^2 t) = e^{2u}.$$

Therefore, by Cramer's rule,

$$J^{-1} = e^{-2u} \begin{vmatrix} e^u \cosh t & -e^u \sinh t \\ -e^u \sinh t & e^u \cosh t \end{vmatrix}.$$

Thus

$$\nabla u = \left(e^{-u} \cosh t, -e^{-u} \sinh t\right), \quad \nabla t = \left(-e^{-u} \sinh t, e^{-u} \cosh t\right).$$

It is easy to check that these have the reciprocal orthonormality properties that they ought to have:

$$\left\langle \frac{\partial \vec{r}}{\partial u}, \nabla u \right\rangle = 1, \quad \left\langle \frac{\partial \vec{r}}{\partial u}, \nabla t \right\rangle = 0, \quad \left\langle \frac{\partial \vec{r}}{\partial t}, \nabla u \right\rangle = 0, \quad \left\langle \frac{\partial \vec{r}}{\partial t}, \nabla t \right\rangle = 1.$$

(c) Calculate $\iint_D xy^2\,dx\,dy$ when D is the region bounded by the curves $u=0$, $u=2$, $t=0$, $t=1$.

$$\int_0^2 du \int_0^1 dt\, xy^2 J = \int_0^2 du \int_0^1 dt\, e^{5u}\cosh t \sinh^2 t$$
$$= \frac{1}{5}\ e^{5u}\Big|_0^2\ \frac{1}{3}\ \sinh^3 t\Big|_0^1 = \frac{1}{15}(e^{10}-1)\sinh^3 1.$$

4. *(15 pts.)* Tell whether each of these formulas defines an *inner product* on the space $\mathcal{C}(0,5)$ (the real-valued continuous functions of t , where $0<t<5$). If not, briefly explain why not.

(a) $$\langle f,g\rangle = \int_0^5 f(t)^2 g(t)^2\,dt$$

NO — not bilinear.

(b) $$\langle f,g\rangle = \int_0^5 \frac{f(t)g(t)}{1+t^2}\,dt$$

YES.

(c) $$\langle f,g\rangle = \int_0^{\pi/2} f(t)g(t)\,dt$$

NO — not positive *definite*: If $f(t)=0$ for $t<\frac{\pi}{2}$ (but f is not zero everywhere in the interval from $\frac{\pi}{2}$ to 5), then $\langle f,f\rangle = 0$ although f is not the zero vector!

5. *(15 pts.)* Find an orthonormal basis for $\mathbf{R}^3$ whose first element is $\hat{u}_1 = \frac{1}{\sqrt{6}}(1,1,2)$.

Note that $\hat{u}_1$ has norm one, so we can put it into the basis unchanged. We continue by the Gram–Schmidt procedure. Choose *any* vector linearly independent of $\hat{u}_1$ to be $\vec{v}_2$; for example, $\vec{v}_2 = (1,0,0)$. Its projection onto $\hat{u}_1$ is

$$\vec{v}_\parallel = (\hat{u}_1\cdot\vec{v}_2)\hat{u}_1 = \frac{1}{6}(1+0+0)(1,1,2) = \frac{1}{6}(1,1,2).$$

So the perpendicular part is

$$\vec{v}_\perp = (1,0,0) - \frac{1}{6}(1,1,2) = \frac{1}{6}(5,-1,-2).$$

Since $\sqrt{25+1+4} = \sqrt{30}$, the normalized vector in this direction is

$$\hat{u}_2 = \frac{1}{\sqrt{30}}(5,-1,-2).$$

Now we need to find a unit vector perpendicular to the two we've found so far.

Method 1: $\hat{u}_3 \equiv \hat{u}_1 \times \hat{u}_2 =$

$$\frac{1}{\sqrt{6\cdot 30}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 2 \\ 5 & -1 & -2 \end{vmatrix} = \frac{1}{6\sqrt{5}}(0\,\hat{\imath} + 12\,\hat{\jmath} - 6\,\hat{k}) = \frac{1}{\sqrt{5}}(0,2,-1).$$

Method 2: Let $\vec{v}_3 = (0,1,0)$. Its projection onto the plane of the first two vectors is

$$\vec{v}_{\|} = (\hat{u}_1 \cdot \vec{v}_3)\hat{u}_1 + (\hat{u}_2 \cdot \vec{v}_3)\hat{u}_2 = \frac{1}{6}(1,1,2) + \frac{-1}{30}(5,-1,-2).$$

So

$$\vec{v}_{\perp} = \left(-\tfrac{1}{6} + \tfrac{5}{30}, 1 - \tfrac{1}{6} - \tfrac{1}{30}, -\tfrac{2}{6} - \tfrac{2}{30}\right) = \left(0, \tfrac{4}{5}, -\tfrac{2}{5}\right),$$

which normalizes to the same $\hat{u}_3$ as before.