1.4 Curves and Tangent Vectors

Vectors are associated not only with straight lines but also with curved lines. A curve can be described by a vector-valued function of a real variable,

$$f\colon \mathbf{R} \to \mathbf{R}^p.$$

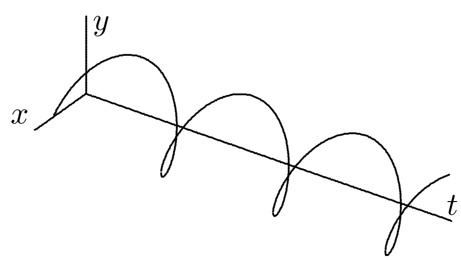
Or, to look at the situation from the other way around, a function from $\mathbf{R}$ into $\mathbf{R}^p$ can be represented geometrically by a curve. (The notation $f\colon \mathbf{R} \to \mathbf{R}^p$ means that f is a function that takes elements of $\mathbf{R}$ as input and yields elements of $\mathbf{R}^p$ as output.)

In fact, there are two different ways in which we can visualize such a function as a curve.

1. We can *graph* the function in a space of dimension $p+1$. For example, if $p = 2$ and

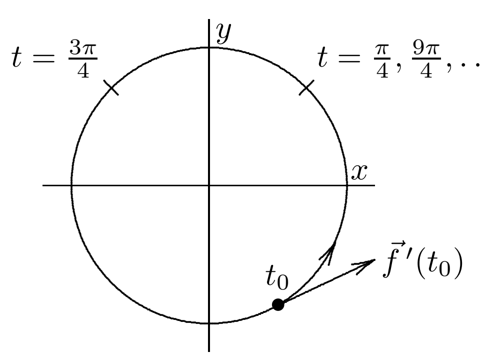
   $$\vec f(t) = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{where} \qquad x = \cos t, \quad y = \sin t,$$

   then (for a certain orientation of the axes) the graph looks like this:

   This graph is a *helix*. With modern computer software it is not hard to produce a genuinely three-dimensional image of the curve, which can be rotated on the computer screen to reveal the curve's geometrical nature more clearly than a single two-dimensional projection on the printed page can do. But if we insist on visualizing functions this way as p increases, we will quickly run out of dimensions.

2. We can represent the function as a *parametrized curve* in p-dimensional space. That is, for each value of the independent variable, t, we plot the point $\vec f(t)$ in $\mathbf{R}^p$. For the previous example the curve is a circle:

We can think of each point as being labeled by the value of t that maps into it, but note that there could be more than one such value. A 3-dimensional example is

$$\vec{g}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \text{where} \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

The curve in this case is the same helix as before, with the t-axis relabeled as z-axis. Only its interpretation has changed, and it is important to understand the conceptual difference. In one case there are 3 variables, in the other there are 4 (three dependent and one independent).

We can define the *derivative* of a vector-valued function by taking the ordinary derivative of each of its coordinates:

$$\vec{f}\,'(t) = \begin{pmatrix} f_1'(t) \\ f_2'(t) \\ \vdots \\ f_p'(t) \end{pmatrix}.$$

(A more profound definition will come later.) For our circle,

$$\vec{f}\,'(t) = \begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

If t has the interpretation of time and $\vec{x} = \vec{f}(t)$ that of position, then $\vec{f}\,'(t)$ is the *velocity* at time t. A geometrical interpretation of $\vec{f}\,'$ is as a *tangent vector* to the parametrized curve, with length proportional to the "speed" at which the curve is traced out by the given parametrization. Customarily one thinks of the vector $\vec{f}\,'(t_0)$ as being attached to the corresponding point, $\vec{x}_0 \equiv \vec{f}(t_0)$, on the curve; see the sketch of the circle above. (Note that this point then becomes the *origin* as far as addition, etc., of vectors of this sort is concerned.) Just as the derivative of an ordinary real-valued function is used to construct the tangent line to the graph of the function, the parametric equation of the *tangent line* to the *graph* of $\vec{f}$ at t_0 is

$$\vec{x} = \vec{f}(t_0) + (t - t_0)\vec{f}\,'(t_0). \tag{1}$$

This line in $\mathbf{R}^{p+1}$ can be thought of as the "best straight-line approximation" to the graph in the small neighborhood near $(t_0, \vec{x}_0)$. One can also think of (1) as a parametric equation of a line in $\mathbf{R}^p$ (through the point $\vec{f}(t_0)$, along the tangent vector $\vec{f}\,'(t_0)$); this is the *tangent line* to the *parametrized curve*.

The numerical significance of the tangent vector is this: When $\vec{f}\,'(t_0)$ is multiplied by a small number $dt \equiv t - t_0$, the result is a vector $d\vec{x}$ that tells approximately how $\vec{f}(t)$ is displaced from $\vec{x}_0$. This is an approximation because the curve is being approximated by its tangent line at $\vec{x}_0$ (or, because the graph of $\vec{f}$ is being approximated by *its* tangent line at (t_0, x_0)).

If $\vec{f}\,'(t_0)$ happens to be the zero vector, then (1) does not define a line. However, it is still true that (1) tells approximately how $\vec{f}(t)$ changes as t moves slightly away from t_0. (That is, $\vec{f}(t)$ is approximately constant in that case!) The fact that (1) is not a line does not necessarily mean that the curve, as a geometrical object in $\mathbf{R}^p$, does not have a tangent line at that point; see Exercises 1.4.4 and 1.4.5.

Later we will see how to generalize all these considerations when the *independent* variable of the function is also multidimensional (see Secs. 2.4 and 3.3–5). A different kind of generalization is to functions whose values $\vec{f}(t)$ lie not in $\mathbf{R}^p$ but in some more general space of vectors, such as those in Examples 1, 3, and 4 of Section 1.1. We'll return to this topic in Sec. 6.3 after building up enough background concepts.

Exercises

1.4.1 Calculate the derivative $\vec{g}\,'(t)$ for the helical curve in the text. Use it to find a parametric representation for the tangent line to the curve at the point where $t = \frac{\pi}{3}$.

1.4.2 Construct the tangent line at $t = \frac{\pi}{3}$ to the circular curve in the text ($x = \cos t, \quad y = \sin t$). What is the relationship between this line and the one in the previous exercise?

1.4.3 A particle is forced to move along the trajectory $\vec{h}(t) = (t^2, 1 + 3t, e^{2t})$. At time $t = 2$ the particle is released from the curved track, and therefore moves off along the tangent line at the constant velocity $\vec{h}'(2)$. Where is the particle at time $t = 3$?

1.4.4 Consider the curve $\beta(t) = (t^5, t^3)$ in $\mathbf{R}^2$.

(a) Show that at the point where $t = 0$, the equations of this section define a tangent vector but not a tangent line.

(b) Find a reparametrization of the curve (define a new variable $\tau = \rho(t)$ via some increasing function ρ) that enables the tangent line at the origin to be constructed in the usual way.

1.4.5 Consider the curve $\beta(t) = (t^2, t^3)$ in $\mathbf{R}^2$.

(a) Show that at the point where $t = 0$, the equations of this section define a tangent vector but not a tangent line.

(b) Show that this curve does not have a tangent line at the origin.

1.4.6 Consider the differential equation $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$, where ω is a parameter (independent of t), with initial data $y(0) = 1$, $y'(0) = -2$.

(a) Find the solution, $y(t)$. (Assume that ω is real and positive.) As ω varies, the solution moves along a curve in an infinite-dimensional space of functions. (Think of each function $y(t)$ as a single point on this curve. Keep in mind that the parameter along the curve is ω, not t.)

(b) Find the derivative of the solution with respect to ω at $\omega = 2$. This function plays the role of tangent vector to the curve of solutions.

(c) Use the result of (b) to construct an approximation to the function y when $\omega = 2.15$. This is a point on the tangent line to the curve of solutions at the point labeled by $\omega = 2$.

(d) Appraise the accuracy of the approximation you got in (c). (You can use a computer or a graphing calculator to plot the exact and approximate solutions as functions of t.) Notice the difference between what happens at small $|t|$ and at large $|t|$.