Final Examination – Solutions

Name: _____

Calculators may be used for simple arithmetic operations only!

1. *(28 pts.)* The eigenvalues of $N = \begin{pmatrix} -1 & -3 & 0 \\ -3 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ are 2 and -4.

(a) If $\nabla f(\vec{0}) = \vec{0}$ and N is the Hessian (second-derivative) matrix of f at $\vec{0}$, is that point a maximum, a minimum, or a saddle point of f?

A saddle point, because the eigenvalues have different signs.

(b) Find an orthonormal basis of eigenvectors of N.

For $\lambda = 2$:
$$\begin{pmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow z \text{ is arbitrary}, \quad x + y = 0.$$

Two orthonormal eigenvectors are $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$

For $\lambda = -4$:
$$\begin{pmatrix} 3 & -3 & 0 \\ -3 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow z = 0, \quad x - y = 0.$$

A normalized eigenvector is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$. It is automatically orthogonal to the other two.

These three eigenvectors make up an orthonormal basis.

(c) Find an orthogonal matrix U and a diagonal matrix D so that $N = UDU^{-1}$ **or** $N = U^{-1}DU$. **State WHICH of these two formulas applies!**

$$D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The question does not demand that U^{-1} be written down, but it is trivial to do so, because it is the transpose of U:

$$U^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since U maps coordinates with respect to the eigenbasis into coordinates with respect to the natural basis, the correct formula is $N = UDU^{-1}$, and with an extra sheet of paper one can check that this is right.

There are many other correct answers. The sign of the second eigenvector could be changed (yielding a U that is its own transpose and inverse). The order of the eigenvalues on the diagonal could be changed. One must be careful to keep each complete solution internally consistent, however.

2. *(24 pts.)* Determine whether each set is linearly independent. If not, find a linearly independent set with the same span.

   (a) $\{(0, 1, 0), (1, 1, 1), (1, 2, 3)\}$

   Independent, because $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2 \neq 0$ (or because the matrix row-reduces to the identity).

   (b) $\{(1, 1, -2, 3), (1, 4, 2, 1), (1, 7, 6, -1)\}$

   Dependent:

   $$\begin{pmatrix} 1 & 1 & -2 & 3 \\ 1 & 4 & 2 & 1 \\ 1 & 7 & 6 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & 4 & -2 \\ 0 & 6 & 8 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & 3 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   The two nonzero rows are a basis for the span:

   $$\{(1, 1, -2, 3), (0, 3, 4, -2)\}.$$

   (In fact, any two of the original rows would also do.)

   (c) $\{1, e^x, e^{2x}\}$

   Independent (by common knowledge).

   (d) $\{1, e^x, 2e^{-x}, \cosh x, \sinh x - 4 \cosh x\}$

   Dependent, because the last two elements can be expressed (linearly) in terms of e^x and e^{-x}. Suitable bases for the span include

   $$\{1, e^x, 2e^{-x}\}, \qquad \{1, e^x, e^{-x}\}, \qquad \text{or} \quad \{1, \cosh x, \sinh x\}.$$

3. *(18 pts.)* Determine whether each formula defines an *inner product* on $\mathbf{R}^3$. If not, explain why not. ($\vec{r}_1 = (x_1, y_1, z_1)$, etc.)

   (a) $\langle \vec{r}_1, \vec{r}_2 \rangle = x_1 x_2 + 2 y_1 y_2 + 3 z_1 z_2$

   This is an inner product (bilinear, symmetric, positive definite).

   (b) $\langle \vec{r}_1, \vec{r}_2 \rangle = x_1 x_2 + y_1 z_2 - z_1 y_2$

   No — not symmetric. (It's also not positive definite: if $x = 0$, then $\langle \vec{r}, \vec{r} \rangle = yz - zy = 0$.)

   (c) $\langle \vec{r}_1, \vec{r}_2 \rangle = x_1 x_2 + y_1 z_2 + z_1 y_2$

   No — not positive. If $x = 0$ and $z = -y$, then $\langle \vec{r}, \vec{r} \rangle = -2y^2 < 0$.

4. *(25 pts.)* Suppose that $x^2 + yz - z^2 = 0$ and $x = s \cosh t$, $y = s \sinh t$.

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at all points where $x = 2$ and $y = 0$.

Differentiate $x^2 + yz - z^2 = 0$ with respect to x and y:

$$2x + y\frac{\partial z}{\partial x} - 2z\frac{\partial z}{\partial x} = 0, \qquad z + y\frac{\partial z}{\partial y} - 2z\frac{\partial z}{\partial y} = 0.$$

Solve:

$$\frac{\partial z}{\partial x} = \frac{2x}{2z - y}, \qquad \frac{\partial z}{\partial y} = \frac{z}{2z - y}.$$

When $x = 2$ and $y = 0$, we have $4 = z^2$, or $z = \pm 2$. Then $\frac{\partial z}{\partial x} = \pm 1$, $\frac{\partial z}{\partial y} = \frac{1}{2}$.

(b) Find the best affine approximation to x and y (as functions of s and t) in the neighborhood of the point where $s = 2$ and $t = 0$.

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cosh t & s \sinh t \\ \sinh t & s \cosh t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{at the point.}$$

At the point, $x = 2$ and $y = 0$. So

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} s - 2 \\ t - 0 \end{pmatrix},$$

or $x \approx 2 + (s - 2) = s$, $y \approx 2t$.

(c) Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ when $s = 2$ and $t = 0$. (Use the results from (a) and (b).)

Chain rule:

$$\begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \pm 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \pm 1 & 1 \end{pmatrix},$$

where the sign $\pm$ is the sign of z. Or, in more classical notation,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \pm 1 + 0 = \pm 1, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = 0 + \frac{1}{2} \cdot 2 = 1.$$

5. *(25 pts.)*   (a)   Explain what a *subspace* is.

A subspace of a vector space is a (nonempty) subset that is closed under addition and scalar multiplication. That is, $\{\vec{0}\} \neq \mathcal{S} \subseteq \mathcal{V}$, and, for all $\vec{v}_1$ and $\vec{v}_2$ in $\mathcal{S}$ and all numbers r, one has $r\vec{v}_1 + \vec{v}_2 \in \mathcal{S}$.

(b) Do the solutions of $x^2 + 2y - z = 3$ form a subspace of $\mathbf{R}^3$?

No — this is a nonlinear equation, and the solution set is not closed under addition or scalar multiplication.

(c) Do the solutions of $x + 5y + 2z = 0$ form a subspace of $\mathbf{R}^3$?

Yes — this is a homogeneous linear equation. (The solutions are the kernel of a linear function.)

(d) Do the solutions of $\dfrac{d^2y}{dt^2} + t\,\dfrac{dy}{dt} + 4y = 0$ form a subspace of $\mathcal{C}^2(-\infty, \infty)$?

Yes — same remarks as for (c). (Solutions of a second-order ordinary differential equation with smooth coefficients will be smooth, certainly in $\mathcal{C}^2$.)

6. *(20 pts.)*   (a) Show, for vector fields $\vec{A}(\vec{x})$ and $\vec{B}(\vec{x})$, that

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}.$$

Method 1: Clearly there will be two sorts of terms, one with derivatives of A and one with derivatives of B . The algebraic structure of each set of terms can be deduced from the antisymmetry and cyclic symmetry of the scalar triple product:

$$\vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A}) \quad \text{and} \quad \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{A} \cdot (\vec{B} \times \vec{C}) = -\vec{A} \cdot (\vec{C} \times \vec{B}).$$

Replacing $\vec{C}$ by ∇ , we get the two terms claimed.

Method 2 (brute force):

$$\begin{aligned}
\nabla \cdot (\vec{A} \times \vec{B}) &= \frac{\partial}{\partial x}(A_y B_z - B_y A_z) + \frac{\partial}{\partial y}(A_z B_x - B_z A_x) + \frac{\partial}{\partial z}(A_x B_y - B_x A_y) \\
&= B_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + B_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + B_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \\
&\quad - \text{(similar terms with } A \text{ and } B \text{ interchanged)} \\
&= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}).
\end{aligned}$$

(b) Show, for scalar functions $f(\vec{x})$ and $g(\vec{x})$ and a closed surface S , that

$$\iint_S (\nabla f \times \nabla g) \cdot d\vec{S} = 0.$$

By Gauss's theorem, the integral is

$$\iiint_V \nabla \cdot (\nabla f \times \nabla g)\, dx\, dy\, dz,$$

where V is the interior of S (i.e., S is the boundary of V). The integrand can be evaluated by (a), and since the curl of a gradient is always zero, the integrand vanishes. Therefore, the original integral is 0 .

7. *(10 pts.)* The *normalized Legendre polynomials* are defined by applying the Gram–Schmidt algorithm with respect to the inner product $\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)\, dt$. The first two of these polynomials are $p_0(t) = \sqrt{\frac{1}{2}}$, $p_1(t) = \sqrt{\frac{3}{2}}\, t$. What is the third one?

$$\langle p_0, t^2 \rangle = \int_{-1}^{1} \sqrt{\frac{1}{2}}\, t^2 \, dt = \frac{\sqrt{2}}{3}\,,$$

$$\langle p_1, t^2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}}\, t^3 \, dt = 0\,.$$

Therefore, the part of t^2 perpendicular to $\mathrm{span}(p_0, p_1)$ is $t^2 - \frac{\sqrt{2}}{3} p_0 = t^2 - \frac{1}{3}$. The normalization integral is

$$\int_{-1}^{1} \left(t^4 - \frac{2}{3} t^2 + \frac{1}{9} \right) dt = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}\,.$$

Thus

$$p_2(t) = \sqrt{\frac{45}{8}} \left(t^2 - \frac{1}{3} \right).$$

(This is also worked out on p. 288 of the textbook.)

8. *(25 pts.)*   (a)   Find all the eigenvalues and eigenvectors of $M = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$.

$$\det(M - \lambda) = \begin{vmatrix} 3 - \lambda & 4 \\ 1 & 6 - \lambda \end{vmatrix} = \lambda^2 - 9\lambda + 14 = (\lambda - 2)(\lambda - 7)\,.$$

The eigenvalues are 2 and 7.

Find the eigenvector for $\lambda = 2$:

$$\begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \Rightarrow x = -4y\,.$$

A suitable eigenvector is $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. (There is no need to normalize it, since we don't expect this eigenbasis to be orthogonal anyway.)

Find the eigenvector for $\lambda = 7$:

$$\begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x = y\,.$$

A suitable eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Find the matrix-valued function e^{tM} that solves the initial-value problem for the differential-equation system $\dfrac{dx}{dt} = 3x + 4y$, $\dfrac{dy}{dt} = x + 6y$.

Method 1: In the coordinates relative to the eigenbasis, the solution operator is

$$e^{tD} = \exp\begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} t = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{7t} \end{pmatrix}.$$

The operators of the coordinate transformation $M = UDU^{-1}$ are

$$U = \begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}, \qquad U^{-1} = \frac{1}{-5}\begin{pmatrix} 1 & -1 \\ -1 & -4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}.$$

Therefore, in the natural coordinate system the solution operator is

$$e^{tM} = Ue^{tD}U^{-1} = \frac{1}{5}\begin{pmatrix} -4 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{7t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 4e^{2t} + e^{7t} & -4e^{2t} + 4e^{7t} \\ -e^{2t} + e^{7t} & e^{2t} + 4e^{7t} \end{pmatrix}.$$

Method 2: In vectorial notation, the most general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}.$$

It is necessary to solve for c_1 and c_2 in terms of the initial data, $x(0)$ and $y(0)$:

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} -4 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The solution of this is

$$c_1 = -\frac{1}{5}x(0) + \frac{1}{5}y(0), \qquad c_2 = \frac{1}{5}x(0) + \frac{4}{5}y(0).$$

(This is equivalent to finding U^{-1} in the other method.) Substitute into the solution and combine terms:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 4e^{2t} + e^{7t} \\ -e^{2t} + e^{7t} \end{pmatrix} x(0) + \frac{1}{5}\begin{pmatrix} -4e^{2t} + 4e^{7t} \\ e^{2t} + 4e^{7t} \end{pmatrix} y(0) = e^{tM}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

with the same matrix e^{tM} as above.

9. *(25 pts.)* Derive the numerical integration formula (quadrature rule) of the form

$$\int_0^e f(x) \ln x \, dx = c_1 f(0) + c_2 f(1) + c_3 f(e)$$

that gives the exact answer whenever f is a quadratic polynomial ($f \in \mathcal{P}_2$). (Note that the integral, although improper, is convergent for any continuous f.) *Hint:*

$$\int x^n \ln x \, dx = x^{n+1} \left[\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} \right] \qquad \text{for } n = 0, 1, 2, \ldots.$$

$$c_1 + c_2 + c_3 = \int_0^e \ln x \, dx = 0,$$

$$c_2 + e c_3 = \int_0^e x \ln x \, dx = \frac{e^2}{4},$$

$$c_2 + e^2 c_3 = \int_0^e x^2 \ln x \, dx = \frac{2e^3}{9}.$$

The easiest way to solve the system is Cramer's rule. The determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & e \\ 0 & 1 & e^2 \end{vmatrix} = e^2 - e.$$

$$c_1 = \frac{1}{e^2 - e} \begin{vmatrix} 0 & 1 & 1 \\ \frac{e^2}{4} & 1 & e \\ \frac{2e^3}{9} & 1 & e^2 \end{vmatrix} = \frac{1}{e^2 - e} \left[- \begin{vmatrix} \frac{e^2}{4} & e \\ \frac{2e^3}{9} & e^2 \end{vmatrix} + \begin{vmatrix} \frac{e^2}{4} & 1 \\ \frac{2e^3}{9} & 1 \end{vmatrix} \right]$$

$$= \frac{1}{e^2 - e} \left(-\frac{e^4}{4} + \frac{2e^4}{9} + \frac{e^2}{4} - \frac{2e^3}{9} \right) = \frac{-\frac{e^3}{36} - \frac{2e^2}{9} + \frac{e}{4}}{e - 1}.$$

$$c_2 = \frac{1}{e^2 - e} \begin{vmatrix} 1 & 0 & 1 \\ 0 & \frac{e^2}{4} & e \\ 0 & \frac{2e^3}{9} & e^2 \end{vmatrix} = \frac{1}{e^2 - e} \left(\frac{e^4}{4} - \frac{2e^4}{9} \right) = \frac{\frac{e^3}{36}}{e - 1}.$$

$$c_3 = \frac{1}{e^2 - e} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & \frac{e^2}{4} \\ 0 & 1 & \frac{2e^3}{9} \end{vmatrix} = \frac{1}{e^2 - e} \left(\frac{2e^3}{9} - \frac{e^2}{4} \right) = \frac{\frac{2e^2}{9} - \frac{e}{4}}{e - 1}.$$

Addendum: One student showed that the solution by row reduction not only is feasible, it reveals that the numerator in c_1 can be factored to cancel the denominator: $c_1 = -\frac{e^2}{36} - \frac{e}{4}$.