Math. 311 (Fulling) 19 April 2002

Test C – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(24 pts.)* Let $\vec{F}(\vec{r}) = (e^x + y)\hat{\imath} + y^3\hat{\jmath} + (z^2 - x)\hat{k}$.

(a) Calculate $\nabla \cdot \vec{F}$.

$$e^x + 3y^2 + 2z.$$

(b) Calculate $\nabla \times \vec{F}$.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ e^x + y & y^3 & z^2 - x \end{vmatrix} = 0\,\hat{\imath} + (-1)(-\hat{\jmath}) + (-1)\hat{k} = \hat{\jmath} - \hat{k}.$$

(c) By the method of your choice, calculate $\oint \vec{F} \cdot d\vec{r}$ around the rectangular path from the origin, to $(1,0,1)$, to $(1,1,1)$, to $(0,1,0)$, and back to $(0,0,0)$. (This path lies in the plane where $z = x$ with y arbitrary.)

Method 1: Apply the Stokes theorem to the result of (b).

$$\iint 0\,dy\,dz + \iint 1\,dz\,dx + \iint (-1)dx\,dy = 0 + 0 + \int_0^1 dx \int_0^1 dy(-1) = -1.$$

Variant of Method 1: The upward unit normal to the plane is $\hat{n} = \frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{k})$. Thus $\hat{n}\cdot\nabla\times\vec{F} = -\frac{1}{\sqrt{2}}$. Multiply by the area of the rectangle, $\sqrt{2}$, to get -1 .

Method 2: Note that y is constant on the first and third sides of the rectangle, while *only* y varies on the other two sides. So the line integral can be written out as

$$\left(\int_0^1 (e^x + 0)\,dx + \int_0^1 (z^2 - z)\,dz\right) + \int_0^1 y^3\,dy + \left(\int_1^0 (e^x + 1)\,dx + \int_1^0 (z^2 - z)\,dz\right) + \int_1^0 y^3\,dy.$$

Almost everything cancels! We are left with

$$-\int_0^1 dx = -1.$$

2. *(18 pts.)* Find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by

$$\{\vec{v}_1 = (2,0,0,0), \quad \vec{v}_2 = (1,1,0,1), \quad \vec{v}_3 = (-2,1,1,0)\}.$$

$$\hat{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = (1,0,0,0).$$

$$v_{2\|} = (\hat{u}_1 \cdot \vec{v}_2)\hat{u}_1 = \hat{u}_1 .$$

$$v_{2\perp} = \vec{v}_2 - \hat{u}_1 = (0,1,0,1).$$

$$\hat{u}_2 = \frac{\vec{v}_{2\perp}}{\|\vec{v}_{2\perp}\|} = \frac{1}{\sqrt{2}}(0,1,0,1).$$

$$v_{3\|} = (\hat{u}_1 \cdot \vec{v}_3)\hat{u}_1 + (\hat{u}_2 \cdot \vec{v}_3)\hat{u}_2 = -2\hat{u}_1 + \tfrac{1}{2}(1)(0,1,0,1) = (-2, \tfrac{1}{2}, 0, \tfrac{1}{2}).$$

$$v_{3\perp} = (-2,1,1,0) - (-2, \tfrac{1}{2}, 0, \tfrac{1}{2}) = (0, \tfrac{1}{2}, 1, -\tfrac{1}{2}).$$

$$\|v_{3\perp}\|^2 = \tfrac{1}{4} + 1 + \tfrac{1}{4} = \tfrac{3}{2}.$$

$$\hat{u}_3 = \sqrt{\tfrac{2}{3}}\, v_{3\perp} = \frac{1}{\sqrt{6}}(0,1,2,-1).$$

3. *(15 pts.)* Let $\vec{A}(\vec{r}) = x^3\hat{\imath} + y^3\hat{\jmath} + z^3\hat{k}$, and let S be the hemispherical surface $r = 2$, $0 \le \theta < \pi$, $0 \le \phi < \pi$ (i.e., the part of the sphere of radius 2 that lies in the $y > 0$ half-space). Calculate $\iint_S \vec{A} \cdot d\vec{S}$ by the method of your choice.

Note first that $\nabla \cdot \vec{A} = 3x^2 + 3y^2 + 3z^2 = 3r^2$. Since the divergence is a rather simple function, it would be nice to be able to use Gauss's theorem. We could close off the hemisphere by the $y = 0$ plane. The flux through that plane is 0 (since $\hat{n} = \hat{\jmath}$ and $y^3 = 0$), so the integral we want is just the integral of $3r^2$ over the half ball (solid hemisphere). [*Alternative argument:* By symmetry, the flux through the other hemispherical surface is equal to the the flux we want, so we can integrate $3r^2$ over the entire ball and divide by 2.]

$$\int_0^2 r^2\, dr \int_0^\pi \sin\theta\, d\theta \int_0^\pi d\phi\, 3r^2 = 2\pi \int_0^2 3r^4\, dr = \frac{6\pi}{5} r^5 \Big|_0^2 = \frac{192\pi}{5}.$$

Brief look at direct methods: One must resist the temptation to write $\vec{A} = r^3\hat{r}$ — that is false! In polar coordinates,

$$\hat{n} = \hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{z}.$$

So

$$\vec{A}\cdot\hat{n} = r^3\sin^4\theta\cos^4\phi + r^3\sin^4\theta\sin^4\phi + r^3\cos^4\theta, \qquad dS = r^2\sin\theta\, d\theta\, d\phi, \qquad r = 2.$$

Thus the integral equals $I_x + I_y + I_z$, where

$$I_x = 2^5 \int_0^\pi d\phi \int_0^\pi d\theta\, \sin^5\theta\cos^4\phi, \qquad I_y = 2^5 \int_0^\pi d\phi \int_0^\pi d\theta\, \sin^5\theta\sin^4\phi,$$

$$I_z = 2^5 \int_0^\pi d\phi \int_0^\pi d\theta\, \cos^4\theta\sin\theta.$$

(The parametric method of pp. 385–386 will lead to the same integrals.) These integrals are standard but a bit unpleasant, except for $I_z = 64\pi/5$. The easiest way to do the other two is to note that by geometrical symmetry $I_x = I_z$ and by translational symmetry of the trig functions $I_y = I_x$. Thus all three are equal and we get the Gauss result again.

4. *(28 pts.)* Define a new coordinate system in a region of the x–y plane by

$$x = u + \tfrac{1}{3}v^3, \qquad y = e^v.$$

(a) Find the formulas for the tangent vectors to the coordinate curves.

$$\frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} v^2 \\ e^v \end{pmatrix}.$$

For future reference, let's put these together into a Jacobian matrix,

$$J = \begin{pmatrix} 1 & v^2 \\ 0 & e^v \end{pmatrix}.$$

(b) Find the area of the region R bounded by the curves $u = 0$, $u = 3$, $v = -2$, and $v = 1$.

We have

$$\det J = e^v,$$

so the area is

$$\int_0^3 du \int_{-2}^1 e^v \, dv = 3(e - e^{-2}).$$

(c) Find the formulas (in terms of u and v) for the normal vectors to the coordinate curves.

We find

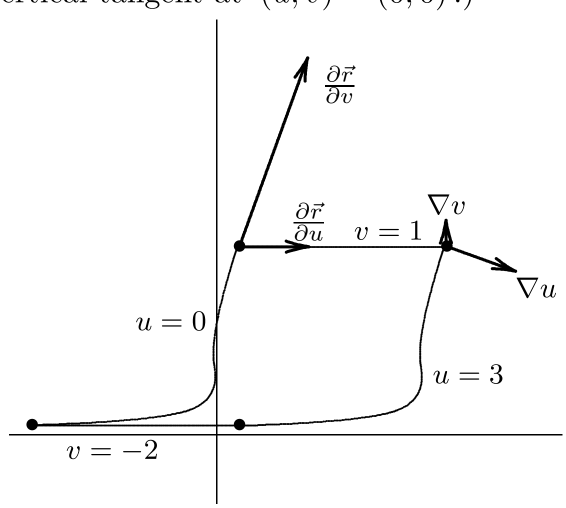
$$J^{-1} = e^{-v} \begin{pmatrix} e^v & -v^2 \\ 0 & 1 \end{pmatrix}.$$

The normal vectors are the rows of this matrix,

$$\nabla u = (1, -v^2 e^{-v}), \qquad \nabla v = (0, e^{-v}).$$

(d) Do **ONE** of these *(10 pts. extra credit for both)*: [Continue on next page if necessary.]

A. In the x–y plane sketch the region R, the basis of tangent vectors at the point where $(u, v) = (0, 1)$, and the basis of normal vectors at the point where $(u, v) = (3, 1)$. (Graphing calculators are allowed. Note that the curve $u = 0$ has an inflection and vertical tangent at $(u, v) = (0, 0)$.)



B. Discuss the "global" properties of this coordinate transformation: Does it cover the whole x–y plane? What is the largest natural range of the variables (u,v)? Is the mapping from (u,v) to (x,y) one-to-one? Does it have a smooth (differentiable) inverse?

Since $\det J$ is never 0, the transformation has a smooth local inverse at every point. Since $y = e^v$ is always positive, the coordinate transformation is defined only in the upper half plane. However, as v ranges from $-\infty$ to $+\infty$, y ranges from 0 to ∞ injectively. Furthermore, for a fixed v, u is uniquely determined by x and vice versa. Thus the natural range of (u,v) is all of $\mathbf{R}^2$, and the coordinate transformation maps this plane bijectively onto the upper half plane $-\infty < x < \infty$, $0 < y < \infty$. (Thus the "local inverse" is actually a global inverse.)

5. *(15 pts.)* Calculate the determinant $\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & \pi & 2 & 3 & 4 \\ 4 & \pi^2 & 3 & 2 & 2 \\ 1 & \pi^3 & 1 & 1 & 1 \\ \pi & \pi^4 & \pi & 2\pi & \pi\sqrt{7} \end{vmatrix}$.

Expand in minors of the top row, and extract a factor from the bottom row:

$$(-\pi)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & \sqrt{7} \end{vmatrix}.$$

Add the first row to the second row:

$$(-\pi)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 5 & 5 & 6 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & \sqrt{7} \end{vmatrix}.$$

Subtract 5 times the third row from the second row:

$$(-\pi)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & \sqrt{7} \end{vmatrix}.$$

Expand in minors of the second row:

$$(-\pi)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = (+\pi)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = \pi\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \pi.$$