Math. 311 (Fulling) 11 May 2004

Final Examination – Solutions

Name: ________________________________

Calculators may be used for simple arithmetic operations only!

1. *(30 pts.)*

(a) Find an **ORTHONORMAL** basis of eigenvectors of $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Eigenvalues:

$$0 = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$$
$$= (1-\lambda)[(2-\lambda)^2 - 1] = (1-\lambda)(\lambda^2 - 4\lambda + 3) = -(\lambda-1)^2(\lambda-3).$$

Eigenvectors with $\lambda = 3$*:* I suppress the columns of zeros at the end.

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} z \text{ arbitrary,} \\ x = z, \\ y = 0. \end{array}$$

A normalized vector is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Eigenvectors with $\lambda = 1$*:*

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} z \text{ arbitrary,} \\ y \text{ arbitrary,} \\ x = -z. \end{array}$$

Two orthogonal and normalized vectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

These three vectors form an ON basis.

(b) If $\nabla f(\vec{v}) = \vec{0}$ and A is the second-derivative matrix of f at $\vec{v}$, does $\vec{v}$ mark a maximum, minimum, or saddle point of f? Explain.

A minimum, because all the eigenvalues are positive.

2. *(20 pts.)* When Test C ended, we had determined these properties of the vector field $\vec{B}(x,y,z) = \dfrac{x}{x^2+y^2}\,\hat{\imath} + \dfrac{y}{x^2+y^2}\,\hat{\jmath}$:

   (1) $\iint_S \vec{B}\cdot d\vec{S} = 3\pi$ when S is the piece of cylindrical surface defined in standard cylindrical coordinates by $r=2$, $0<\theta<\pi$, $0<z<3$ (with outward normal).

   (2) There exists a vector potential $\vec{A}$ such that $\nabla\times\vec{A} = \vec{B}$ everywhere except possibly on the axis, $x=y=0$.

   In fact (see Qu. 10(B) below), one can find such an $\vec{A}$ with the additional properties

   (3) $A_z = 0$ everywhere (i.e., like $\vec{B}$, $\vec{A}$ has no $\hat{k}$ component).

   (4) $\vec{A}(x,y,0) = \vec{0}$ for all x and y.

   (a) Use the Stokes theorem to evaluate $\int_C \vec{A}\cdot d\vec{r}$ where C is the semicircle $r=2$, $z=3$, $0<\theta<\pi$. (Justify answer fully.)

The theorem says that the surface integral in (1) equals the line integral of $\vec{A}$ around the entire boundary of the cylindrical fragment. The integral along the two straight sides is 0 because $\vec{A}\cdot\hat{T} = \pm A_z = 0$ there. The integral around the semicircle at $z=0$ is 0 because $\vec{A}=0$ there. So the integral around the other semicircle gets it all! The line integral in Stokes runs counterclockwise from the viewpoint of the outward normal; it follows (draw a sketch) that the semicircle in question is traversed in the direction of decreasing θ. Therefore, $\displaystyle\int_C \vec{A}\cdot d\vec{r} = -3\pi$.

   (b) Explain why it is **wrong** to argue by Stokes's theorem that the integral $\oint \vec{A}\cdot d\vec{r}$ around the whole circle at $r=2$ and $z=3$ is equal to 0 because the flux of $\vec{B}$ through the disk ($r<2$, $z=3$) is 0.

The condition $\vec{B} = \nabla\times\vec{A}$ is not necessarily satisfied at the center of the disk, so Stokes does not apply to this circle. (See answer to Qu. 10(B) for more information.)

3. *(15 pts.)* Find the volume of the parallelepiped determined by $\{(2,0,2),(3,3,0),(0,1,1)\}$, and discuss the handedness (right or left) of that list of vectors as a basis for $\mathbf{R}^3$.

$$\begin{vmatrix} 2&0&2\\ 3&3&0\\ 0&1&1 \end{vmatrix} = 2\cdot 3\begin{vmatrix} 1&0&1\\ 1&1&0\\ 0&1&1 \end{vmatrix} = 6\begin{vmatrix} 1&0&1\\ 1&1&0\\ -1&1&0 \end{vmatrix} = 6\cdot\begin{vmatrix} 1&1\\ -1&1 \end{vmatrix} = 12.$$

Since the determinant is positive, it is the volume and the basis is right-handed. (Of course, it doesn't matter whether we write the vectors as rows or columns, and there are many other routes to evaluate the determinant.)

4. *(15 pts.)* If three variables are constrained by the equations $a+b+c=1$ and $abc = \frac{1}{36}$, find $\frac{da}{db}$ and $\frac{dc}{db}$ at the point $(a,b,c) = (\frac12,\frac13,\frac16)$.

Differentiate both equations with respect to b:

$$\frac{da}{db} + 1 + \frac{dc}{db} = 0, \qquad \frac{da}{db}\,bc + ac + ab\,\frac{dc}{db} = 0.$$

Insert the values at the point of interest (which does satisfy both constraints, we notice):

$$\frac{da}{db} + \frac{dc}{db} = -1, \qquad \frac{1}{18}\,\frac{da}{db} + \frac{1}{6}\,\frac{dc}{db} = -\frac{1}{12}\,.$$

Multiply the second equation by 18 to get the system

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 3 & -\frac{3}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & -\frac{1}{2} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{4} \\ 0 & 1 & -\frac{1}{4} \end{pmatrix}.$$

Thus $\dfrac{da}{db} = -\dfrac{3}{4}$ and $\dfrac{dc}{db} = -\dfrac{1}{4}$.

5. *(25 pts.)*

(a) Find all solutions of $\begin{cases} x \qquad + z = 0, \\ x + y - z = b, \\ 3x + y + z = c, \end{cases}$ where b and c are arbitrary constants.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & b \\ 3 & 1 & 1 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & b \\ 0 & 1 & -2 & c \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & b \\ 0 & 0 & 0 & c-b \end{pmatrix}.$$

Clearly there is *no solution* if $b \neq c$. If $b = c$, then

$$z \text{ is arbitrary}, \quad y = b + 2z, \quad x = -z.$$

(b) Discuss the kernel, rank, and range of the linear function with matrix $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$.

The kernel is the case $b = c = 0$ of (a):

$$\text{kernel} = \text{span} \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Therefore, the rank (dimension of the range) equals 2 (by the theorem in Qu. 10(A)). There are several ways to determine the range:

Method 1 (traditional): Column-reduce the matrix (i.e., row-reduce its transpose), getting

$$\text{range} = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Method 2: From (a), it is obvious that multiples of $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are in the range. Since the rank is 2, we know that we need another linearly independent vector. Any of the columns of the original matrix (except the middle one), or any linear combination of them, will do. The result is a basis equivalent to the one found by the first method.

Method 3: Redo (a) with the first equation replaced by $x + z = a$. Conclusion: $c = b + 2a$. Then the basis vectors found by the first method fall out from the choices $a = 1$, $b = 0$ and $a = 0$, $b = 1$.

6. *(30 pts.)*

(a) Find all the eigenvalues and eigenvectors of $B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Eigenvalues: $0 = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$. Thus

$$\lambda = \tfrac{1}{2}(4 \pm \sqrt{16+20}) = \tfrac{1}{2}(4 \pm 6) = \begin{cases} 5 \\ -1 \end{cases}.$$

Eigenvectors with $\lambda = 5$:

$$\begin{pmatrix} -4 & 2 & 0 \\ 4 & -2 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = \frac{1}{2}y.$$

The eigenvectors are the multiples of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Eigenvectors with $\lambda = -1$:

$$\begin{pmatrix} 2 & 2 & 0 \\ 4 & 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x = -y.$$

The eigenvectors are the multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(b) Find a diagonal matrix D and a matrix Q such that $B = QDQ^{-1}$ or $D = QBQ^{-1}$. **State WHICH of these two formulas applies!**

Let

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}.$$

Since Q has the eigenvectors as columns, it maps coordinates with respect to the eigenbasis into coordinates with respect to the original or natural basis. Therefore, the correct equation is $B = QDQ^{-1}$ (which you can verify by calculating Q^{-1} and doing the matrix multiplications). *Note:* You can write the eigenvalues in the other order in D, but then your Q must change accordingly. Also, if you normalized your eigenvectors differently, then you got a different Q; in the end the differences are cancelled by the resulting differences in Q^{-1}.

(c) Use D and Q to solve the ODE system $\frac{dx}{dt} = x + 2y$, $\frac{dy}{dt} = 4x + 3y$ with arbitrary initial data $x(0)$, $y(0)$.

Method 1:

$$e^{tD} = \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix};$$

$$e^{tB} = Qe^{tD}Q^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix} \left(\frac{-1}{3}\right) \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$$
$$= \frac{1}{3} \begin{pmatrix} e^{5t} & e^{-t} \\ 2e^{5t} & -e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} e^{5t} + 2e^{-t} & e^{5t} - e^{-t} \\ 2e^{5t} - 2e^{-t} & 2e^{5t} + e^{-t} \end{pmatrix}.$$

Then
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tM} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$
Method 2: The general solution is
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} = \begin{pmatrix} e^{5t} & e^{-t} \\ 2e^{5t} & -e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Qe^{tD} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$
To determine the c_j we look at the initial data:
$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = Qe^0 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Q^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

7. *(20 pts.)* Roger Rapidrudder measured the velocity and pressure of the air to be $\vec{v}(x,y) = x^2\,\hat{\imath} - xy\,\hat{\jmath}$, $P(x,y) = x^3 + y^2$. He defines polar coordinates as usual by $x = r\cos\theta$, $y = r\sin\theta$.

   (a) Use the multivariable chain rule to find the partial derivatives $\dfrac{\partial P}{\partial r}$ and $\dfrac{\partial P}{\partial \theta}$.

$$\left(\tfrac{\partial P}{\partial r}\,,\ \tfrac{\partial p}{\partial \theta} \right) = \nabla P \cdot \left(\tfrac{\partial \vec{r}}{\partial r}\,,\ \tfrac{\partial \vec{r}}{\partial \theta} \right) = (\,3x^2\,,\ 2y\,) \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$
$$= (\,3x^2\cos\theta + 2y\sin\theta\,,\ -3rx^2\sin\theta + 2ry\cos\theta\,).$$
It would be nicer to write this entirely in terms of polars:
$$(\,3r^2\cos^3\theta + 2r\sin^2\theta\,,\ -3r^3\cos^2\theta\sin\theta + 2r^2\cos\theta\sin\theta\,).$$
Side remark: These expressions do not look dimensionally homogeneous. For the pressure formula to make sense, it really should be of the form $P = ax^3 + by^2$, where a has units such as pascals per cubic meter while b is in pascals per square meter. But it is possible, of course, that $a = 1$ and $b = 1$ in some system of units.

   (b) Find the components of $\vec{v}$ with respect to the polar unit vectors $\hat{r}$, $\hat{\theta}$. (Answer should be of the form $\vec{v} = f(r,\theta)\,\hat{r} + g(r,\theta)\,\hat{\theta}$.)

Start by noting that the unit vectors are
$$\hat{r} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \quad \hat{\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$$
(by simple geometry, or by calculating the tangent vectors to the coordinate curves and normalizing them).
Method 1: Since these are unit vectors, the basis is orthonormal and we can calculate the coefficients as dot products:
$$f = \hat{r}\cdot\vec{v} = x^2\cos\theta - xy\sin\theta = r^2\cos^3\theta - r^2\cos\theta\sin^2\theta,$$
$$g = \hat{\theta}\cdot\vec{v} = -x^2\sin\theta - xy\cos\theta = -r^2\cos^2\theta\sin\theta - r^2\cos^2\theta\sin\theta = -2r^2\cos^2\theta\sin\theta.$$
Method 2: The matrix expressing the polar unit vectors in terms of the Cartesian unit vectors is, from above, $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. The matrix expressing polar coordinates of a vector in terms of Cartesian coordinates is the contragredient (inverse of the transpose) of this one. Since the matrix is orthogonal, its contragredient is itself. So
$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x^2 \\ -xy \end{pmatrix},$$
which of course works out the same as above.

8. *(15 pts.)* Prove the identity $\nabla\times(f\nabla g) = (\nabla f)\times(\nabla g)$ (for scalar functions f and g).

First method: Recall the identity $\nabla\times(f\vec{A}) = f\times\vec{A} + f\,\nabla\times\vec{A}$:

$$\nabla\times(f\nabla g) = (\nabla f)\times(\nabla g) + f\,\nabla\times\nabla g = (\nabla f)\times(\nabla g).$$

Second method: It suffices to check one component of the curl, say the $\hat{\imath}$ component:

$$\frac{\partial}{\partial y}\left(f\,\frac{\partial g}{\partial z}\right) - \frac{\partial}{\partial z}\left(f\,\frac{\partial g}{\partial y}\right) = \frac{\partial f}{\partial y}\,\frac{\partial g}{\partial z} + f\,\frac{\partial^2 g}{\partial y\,\partial z} - \frac{\partial f}{\partial z}\,\frac{\partial g}{\partial y} - f\,\frac{\partial^2 g}{\partial z\,\partial y} = \frac{\partial f}{\partial y}\,\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\,\frac{\partial g}{\partial y}\,,$$

which is the $\hat{\imath}$ component of $(\nabla f)\times(\nabla g)$. *Remark:* This argument is not an evil "proof by example" such as your teachers and graders are constantly condemning. Both sides of the asserted identity are expressed in terms of the abstract vector operations, which do not depend on the orientation of coordinate axes in space (are "rotationally covariant"). Therefore, if the calculation works out properly for one component, it must for all (and the other calculations could be obtained by cyclic permutation of the 3 directions).

9. *(15 pts.)* Prove that $\langle f,g\rangle = \int_0^1 [f(t)g(t) + f'(t)g'(t)]\,dt$ is an inner product on the function space $\mathcal{C}^1(0,1)$. (The prime means differentiation.)

(1) It is obviously *symmetric*: $\langle g,f\rangle = \langle f,g\rangle$. (2) It is fairly obviously *bilinear*, since differentiation is linear and each term is of first order in each of the functions. In more detail:

$$\begin{aligned}\langle f, g_1 + rg_2\rangle &= \int_0^1 [f(g_1+rg_2) + f'(g_1'+rg_2')]\,dt \\ &= \int_0^1 [fg_1 + f'g_1']\,dt + r\int_0^1 [fg_2 + f'g_2']\,dt = \langle f,g_1\rangle + r\langle f,g_2\rangle.\end{aligned}$$

(3) It is *positive definite*: $\langle f,f\rangle = \int_0^1 [f(t)^2 + f'(t)^2]\,dt$, and as the integral of a sum of squares, this is nonnegative and is equal to 0 only when both the terms are 0 — i.e., $f(t)=0$ for all t in the interval.

10. *(20 pts.)* Do **ONE** of these; extra credit for doing **TWO**.

(A) Prove the theorem that dim(domain) = dim(kernel) + dim(range) for a linear function with a finite-dimensional domain. *Hint:* Choose or construct bases for the various subspaces involved.

Choose a basis $\{\vec{k}_1,\ldots,\vec{k}_n\}$ for the kernel. Extend it to a basis $\{\vec{k}_1,\ldots,\vec{k}_n,\vec{u}_1,\cdots,\vec{u}_{d-n}\}$ for the entire domain. Here d is the dimension of the domain, and n that of the kernel. Note that since basis vectors are independent, no linear combination of the $\vec{u}_j$ vectors, except the zero vector, is a member of the kernel. Now apply the linear function L, getting the set

$$\{L(\vec{k}_1),\ldots,L(\vec{k}_n),L(\vec{u}_1),\cdots,L(\vec{u}_{d-n})\}$$

in the range. This set spans the range, since by definition every vector in the range is the image under L of one or more vectors in the domain, and all such are linear combinations of the basis vectors. The n vectors $L(\vec{k}_j)$ are equal to 0, by definition of the kernel, so they can be discarded, and the remaining vectors $L(\vec{u}_j)$ still span the range. They also are independent, because

$$\sum_{j=1}^{d-n} c_j L(\vec{u}_j) = 0 \text{ would imply } L\left(\sum_{j=1}^{d-n} c_j\vec{u}_j\right) = 0, \text{ which would say that } \sum_{j=1}^{d-n} c_j\vec{u}_j$$

is in the kernel, a contradiction. Therefore, the range has a basis $\{L(\vec{u}_1),\cdots,L(\vec{u}_{d-n})\}$ of $d-n$ elements, so $d-n$ is the dimension of the range, as was to be proved.

(B) Construct a vector field $\vec{A}(x,y,z)$ with the properties (2), (3), and (4) needed in Qu. 2. Then evaluate directly the two line integrals involved in Qu. 2, parts (a) and (b).

Write out the three components of $\nabla \times \vec{A} = \vec{B}$, with $A_z = 0$:

$$\frac{x}{x^2+y^2} = B_x = -\frac{\partial A_y}{\partial z}\,, \qquad \frac{y}{x^2+y^2} = B_y = \frac{\partial A_x}{\partial z}\,, \qquad 0 = B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\,.$$

Integrate the first two equations, using $A(x,y,0) = \vec{0}$:

$$A_x = \frac{yz}{x^2+y^2} \qquad A_y = \frac{-xz}{x^2+y^2}\,.$$

Check that the third equation is satisfied:

$$\frac{\partial A_x}{\partial y} = \frac{z}{x^2+y^2} - \frac{2y^2 z}{(x^2+y^2)^2} = \frac{(x^2-y^2)z}{(x^2+y^2)^2}\,,$$
$$\frac{\partial A_y}{\partial x} = \frac{-z}{x^2+y^2} + \frac{2x^2 z}{(x^2+y^2)^2} = \frac{(x^2-y^2)z}{(x^2+y^2)^2}\,.$$

To do the line integrals we use θ as parameter and note that $x^2+y^2 = r^2 = 4$ and $z = 3$ everywhere on the curves concerned.

$$\int_C \vec{A}\cdot d\vec{r} = \int \vec{A}\cdot \vec{r}\,'(\theta)\,d\theta = \int_0^\pi \left(\frac{3}{4}\right)[y\,\hat{\imath} - x\,\hat{\jmath}]\cdot[-2\sin\theta\,\hat{\imath} + 2\cos\theta\,\hat{\jmath}]\,d\theta$$
$$= -3\int_0^\pi (\sin^2\theta + \cos^2\theta)\,d\theta = -3\pi.$$

In other words ($\hat{T}$ here being the same as $\hat{\theta}$ in Qu. 7(b)),

$$\vec{A}\cdot\hat{T} = \frac{3}{4}[2\sin\theta\,\hat{\imath} - 2\cos\theta\,\hat{\jmath}]\cdot[-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}] = \frac{3}{4}(-2)(\sin^2\theta + \cos^2\theta) = -\frac{3}{2}\,,$$

and this constant gets multiplied by the arc length of the semicircle, 2π. Because the integrand is constant, it is now clear that the integral around the whole circle is just twice the one along the semicircle, or -6π — not 0.

(C) Find the first two orthogonal polynomials for the inner product in Qu. 9.

Normalize the first vector:

$$\langle 1,1\rangle = \int_0^1 (1+0)\,dt = 1 \;\Rightarrow\; u_0(t) = 1.$$

Find the "parallel part" of the second vector:

$$\langle 1,t\rangle = \int_0^1 (t+0)\,dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}\,.$$

So the perpendicular part is $v_\perp = t - \frac{1}{2}$. Normalize that:

$$\langle v_\perp, v_\perp\rangle = \int_0^1 \left[\left(t-\frac{1}{2}\right)^2 + 1\right]dt = \int_0^1 \left(t^2 - t + \tfrac{5}{4}\right)dt = \left[\frac{t^3}{3} - \frac{t^2}{2} + \frac{5t}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{5}{4} = \frac{13}{12}\,.$$

Therefore,

$$u_1(t) = \sqrt{\frac{12}{13}}\left(t - \frac{1}{2}\right).$$