# Green Functions: Matrices for Infinite-Dimensional Operators

## I. Preliminary remarks

The point of this lecture is to show how certain ideas and facts of finite-dimensional linear algebra partly persist into function spaces (and partly don't).

Typical vector spaces of functions are $\mathcal{C}^n(0,\pi)$. We have seen

(a) Differential operators: E.g.,

$$L = \frac{d^2}{dx^2} + 3, \qquad L\colon \mathcal{C}^2(0,\pi) \to \mathcal{C}^0(0,\pi).$$

**Note:** This notation means $Ly = y'' + 3y$, not $y'' + 3$.

(b) Integral operators: For a suitable function $G$,

$$Gy(x) = \int_0^\pi G(x,t) f(t)\, dt, \qquad G\colon \mathcal{C}^0(0,\pi) \to \mathcal{C}^0(0,\pi).$$

This lecture focuses on the following ...

**Example:** Let $\omega$ be a positive real number, *not* an integer. Define

$$G_\omega(x,t) = \frac{\sin(\omega x_<)\sin(\omega(x_> - \pi))}{\omega \sin(\omega\pi)}\,,$$

where $x_< = \min(x,t)$, $x_> = \max(x,t)$. (It turns out that $G\colon \mathcal{C}^0(0,\pi) \to \mathcal{C}^2(0,\pi)$ in this case.)

**Remark:** $G(x,t)$ is like a *matrix* for the operator $G$. But there is no function that can act as a matrix for $L$ (or for $I\colon \mathcal{C}^0 \to \mathcal{C}^0$). Unfortunately, the official terminology for the function $G(x,t)$ is: the *integral kernel* of the operator $G$.

## II. The main claim: $G$ is the inverse of $L$

For $\omega > 0$ and not integer, define

$$L = \frac{d^2}{dx^2} + \omega^2.$$

Since solutions of differential equations are not unique until initial or boundary data are imposed, in order for $L$ to *have* an inverse (and still be a *linear* operator) we need to restrict its domain to build in enough homogeneous boundary data: Let

$$\mathcal{D} = \{y \in \mathcal{C}^2\colon y(0) = 0 = y(\pi)\}$$

and henceforth consider $L$ with $\mathcal{D}$ as domain.

**Proposition:** $G$ is the *inverse* of $L$:

$$Ly = f \iff y = Gf.$$

**Verification:** We have to show that

$$y(x) = \int_0^\pi G_\omega(x,t) f(t)\,dt$$

satisfies the differential equation $y'' + \omega^2 y = f$ and the boundary conditions $y(0) = 0 = y(\pi)$. The definition of $G_\omega$ gives

$$y(x) = \int_0^x \frac{\sin(\omega t)\sin(\omega(x-\pi))}{\omega\sin(\omega\pi)}\, f(t)\,dt + \int_x^\pi \frac{\sin(\omega x)\sin(\omega(t-\pi))}{\omega\sin(\omega\pi)}\, f(t)\,dt,$$

from which it is easy to see that the boundary conditions are satisfied.

Differentiate to get

$$\begin{aligned} y'(x) &= \frac{\sin(\omega x)\sin(\omega(x-\pi))}{\omega\sin(\omega\pi)}\, f(x) - \frac{\sin(\omega x)\sin(\omega(x-\pi))}{\omega\sin(\omega\pi)}\, f(x) \\ &\quad + \int_0^x \frac{\sin(\omega t)\cos(\omega(x-\pi))}{\sin(\omega\pi)}\, f(t)\,dt + \int_x^\pi \frac{\cos(\omega x)\sin(\omega(t-\pi))}{\sin(\omega\pi)}\, f(t)\,dt, \end{aligned}$$

and the first two terms cancel. Therefore,

$$\begin{aligned} y''(x) &= \frac{\sin(\omega x)\cos(\omega(x-\pi))}{\omega\sin(\omega\pi)}\, f(x) - \frac{\cos(\omega x)\sin(\omega(x-\pi))}{\omega\sin(\omega\pi)}\, f(x) \\ &\quad - \omega\int_0^x \frac{\sin(\omega t) sin(\omega(x-\pi))}{\sin(\omega\pi)}\, f(t)\,dt - \omega\int_x^\pi \frac{\sin(\omega x)\sin(\omega(t-\pi))}{\sin(\omega\pi)}\, f(t)\,dt. \end{aligned}$$

This time the *second* two terms are precisely $-\omega^2 y(x)$. Combining the first two terms by a trig identity, we get

$$\begin{aligned} y''(x) + \omega^2 y(x) &= \frac{\sin(\omega x - \omega x + \omega\pi)}{\sin(\omega\pi)}\, f(x) \\ &= f(x). \end{aligned}$$

This calculation has verified that $LG = I$ on the domain $\mathcal{C}(0,\pi)$ (i.e., $f \xrightarrow{G} y \xrightarrow{L} f$). To show that $GL = I$ on the domain $\mathcal{C}^2(0,\pi)$ (i.e., $y \xrightarrow{L} f \xrightarrow{G} y$) we need to appeal to the uniqueness theorem for solutions of ordinary differential equations.

## III. Uniqueness — or the lack thereof

First consider a square matrix $M$. If $\lambda \in \mathbf{R}$, the notation $M - \lambda$ means the same thing as $M - \lambda I$. The inverse $(M-\lambda)^{-1}$ exists if (and only if) $\det(M-\lambda) \neq 0$. In that case the nonhomogeneous linear equation $(M-\lambda)\vec{v} = \vec{b}$ has exactly one solution for each $\vec{b}$:

$$\vec{v} = (M-\lambda)^{-1}\vec{b}.$$

If $\det(M-\lambda) = 0$ (which happens for a finite list of roots of the polynomial), then there is a vector $\vec{v}_0$ such that $M\vec{v}_0 = \lambda\vec{v}_0$ (called an *eigenvector*). In that case $(M-\lambda)\vec{v} = \vec{b}$ has infinitely many solutions if it has any at all, because to any particular solution we could add any multiple of $\vec{v}_0$. Our big point is that much the same thing happens with the operator $L$.

**Proposition:**

(a) If $\omega$ is a nonzero integer, then $y'' + \omega^2 y = f$ has *many* solutions satisfying $y(0) = 0 = y(\pi)$ if it has any such solutions at all.

(b) If $\omega$ is not a nonzero integer, then this boundary-value problem has at most *one* solution. (Then our previous proposition shows that there is *exactly* one, and hence $GL = I$.)

**Proof:**

(a) There is an eigenvector: $y_0(x) = \sin(\omega x)$ satisfies $y'' + \omega^2 y = 0$, $y(0) = 0 = y(\pi)$.

(b) If $y_1$ and $y_2$ are solutions, then $y = y_1 - y_2$ must satisfy $y'' + \omega^2 y = 0$, $y(0) = 0 = y(\pi)$. The only solutions of the ODE and the first boundary condition are $y(x) = C\sin(\omega x)$, but then the second boundary condition, $C\sin(\omega\pi) = 0$, can't be satisfied unless $C = 0$; so $y_1$ and $y_2$ are the same.

## IV. Existence — or the lack thereof

We still have a loose end to tie up in the case that $\omega^2$ is an eigenvalue.

**Proposition:** When $\omega$ is a nonzero integer:

(a) If $\int_0^\pi f(x)\sin(\omega x)\,dx = 0$, then a solution of the boundary-value problem exists (but isn't unique).

(b) If $\int_0^\pi f(x)\sin(\omega x)\,dx \neq 0$, then no solution exists (satisfying both boundary conditions as well as the ODE).

Compare the situation with a *symmetric* matrix $M$. (Note: $G_\omega(t,x) = G_\omega(x,t)$, which is the analog of the symmetry of $(M-\lambda)^{-1}$.) In general, $\ker(M-\lambda)$ comprises the vectors orthogonal (perpendicular) to all the rows of $M-\lambda$. When $M$ is symmetric, that's the same as the rows orthogonal to all the *columns* of $M-\lambda$ — i.e., orthogonal to the range of $M-\lambda$. In other words: If $(M-\lambda)\vec{v}_0 = 0$, and if $(M-\lambda)\vec{v} = \vec{b}$ has any solutions $\vec{v}$, then $\vec{b}$ is orthogonal to $\vec{v}_0$ (and conversely). The condition in our proposition states that $f$ is "perpendicular" to the eigenvector $\sin(\omega x)$.

**Sketch of proof of proposition:** Solve the ODE $y'' + \omega^2 y = f$ by *variation of parameters*:

$$y(x) = B(x)\sin(\omega x) + A(x)\cos(\omega x).$$

You get solvable first-order differential equations for $A$ and $B$. The solution involves two arbitrary constants of integration, $A_0$ and $B_0$, which ought to to be found by imposing the boundary conditions $y(0) = 0 = y(\pi)$. That results in a $2 \times 2$ linear system to be solved for $A_0$ and $B_0$.

(a) If $\omega$ is not a nonzero integer, the system is nonsingular (the solution is unique) and you discover the formula for the Green function (which I pulled out of a hat earlier).

(b) If $\omega$ *is* a nonzero integer, the algebraic system is singular (rank 1); it is *inconsistent* if the orthogonality integral is not zero, and it has nonunique solutions if the integral is zero.