Math. 311 (Fulling) 18 February 2004

Test A – Solutions

Name: ______________________________

Calculators may be used for simple arithmetic operations only!

1. *(15 pts.)* Find all solutions (w, x, y, z) of the system $\left\{\begin{array}{l} w + 2x + \ \ y + \ \ z = 1\,, \\ 3w + 2x - \ \ y - 2z = 0\,. \end{array}\right\}$

Form the augmented matrix and reduce:

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 3 & 2 & -1 & -2 & 0 \end{pmatrix} \xrightarrow{(2)\to(2)-3(1)} \begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & -4 & -4 & -5 & -3 \end{pmatrix} \xrightarrow{(2)\to-\frac14(2)}$$

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & \frac54 & \frac34 \end{pmatrix} \xrightarrow{(1)\to(1)-2(2)} \begin{pmatrix} 1 & 0 & -1 & -\frac32 & -\frac12 \\ 0 & 1 & 1 & \frac54 & \frac34 \end{pmatrix}.$$

Thus

$$w - y - \tfrac32 z = -\tfrac12,$$
$$x + y + \tfrac54 z = \tfrac34.$$

Let

$$y = s, \quad z = t \qquad s \text{ and } t \text{ arbitrary}).$$

Then

$$w = s + \frac32 t - \frac12\,, \quad x = -s - \frac54 t + \frac34$$

is the general solution. It is quickly checked by substituting back into the original equations.

2. *(10 pts.)* Define a mapping T of the function space $\mathcal{C}^2(-\infty,\infty)$ into the function space $\mathcal{C}(-\infty,\infty)$ by

$$[T(f)](z) \equiv f''(z) + z^2 f(z) + \int_0^z e^{-u} f(u)\,du\,.$$

Is T a linear function? Explain.

YES. Let λ be an arbitrary real number and f and g be arbitrary functions in $\mathcal{C}^2$. Then

$$\begin{aligned} T(\lambda f + g)(z) &= (\lambda f + g)''(z) + z^2(\lambda f + g)(z) + \int_0^z e^{-u}(\lambda f(u) + g(u))\,du \\ &= \lambda f''(z) + g''(z) + \lambda z^2 f(z) + z^2 g(z) + \lambda\int_0^z e^{-u} f(u)\,du + \int_0^z e^{-u} g(u)\,du \\ &= \lambda[Tf](z) + [Tg](z) \end{aligned}$$

(where the known linearity of differentiation and integration have been used). Thus T satisfies the definition of linearity.

3. *(30 pts.)* Define $\left\{\begin{array}{l} u = x^3 - 2y + z\,, \\ v = 4x + e^y + z^2\,, \end{array}\right\}$ $\quad \vec r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \vec r_0 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$

(a) Find the direction of most rapid increase of u at the point r_0.

The direction of fastest increase is the direction of the gradient.

$$\nabla u(\vec r_0) = (3x^2, -2, 1)\big|_{\vec r_0} = (12, -2, 1).$$

The unit vector in that direction (for one point extra credit) is

$$\frac{1}{\sqrt{144+4+1}}\begin{pmatrix} 12 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{149}}\begin{pmatrix} 12 \\ -2 \\ 1 \end{pmatrix}.$$

(b) Define $F\colon \mathbf{R}^3 \to \mathbf{R}^2$ by $\begin{pmatrix} u \\ v \end{pmatrix} = F(\vec r)$. Construct the best affine approximation (a.k.a. the first-order approximation) to F around $\vec r_0$.

The Jacobian matrix of this function is

$$JF = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} 3x^2 & -2 & 1 \\ 4 & e^y & 2z \end{pmatrix}.$$

Evaluate it at $\vec r_0$:

$$J_{\vec r_0}F = \begin{pmatrix} 12 & -2 & 1 \\ 4 & e & 4 \end{pmatrix}.$$

Now

$$F(\vec r) \approx F(\vec r_0) + d_{\vec r_0}F(\vec r - \vec r_0),$$

where the matrix of the differential $d_{\vec r_0}F$ is $J_{\vec r_0}F$. That is,

$$F(r) \approx \begin{pmatrix} 8 \\ 12 + e \end{pmatrix} + \begin{pmatrix} 12 & -2 & 1 \\ 4 & e & 4 \end{pmatrix}\begin{pmatrix} x-2 \\ y-1 \\ z-2 \end{pmatrix}.$$

Further simplification is optional (cf. part (c)).

(c) Define a curve by $\vec r(t) = \begin{pmatrix} 2t^2 \\ t \\ -2\cos(\pi t) \end{pmatrix}$. Note that $\vec r(1) = \vec r_0$. Find the tangent vector to the curve at that point, **and** the parametrized equation of the tangent line.

The tangent vector is

$$\vec r\,'(1) = \begin{pmatrix} 4t \\ 1 \\ 2\pi\sin(\pi t) \end{pmatrix}\Bigg|_{t=1} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}.$$

So the tangent line is

$$\begin{aligned}\vec{r} &= \vec{r}(1) + \vec{r}\,'(1)(t-1) \\ &= \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}(t-1) \\ &= \begin{pmatrix} 2 + 4(t-1) \\ 1 + (t-1) \\ 2 \end{pmatrix}.\end{aligned}$$

Further simplification is possible but not recommended.

(d) Find $\dfrac{d}{dt}F\big(\vec{r}(t)\big)$ at $t = 1$.

We can use parts of (b) and (c) in the chain rule:

$$\begin{aligned}\frac{d}{dt}F\big(\vec{r}(t)\big)\Big|_{t=1} &= d_{\vec{r}_0}F(\vec{r}\,'(1)) \quad \text{[also written } \big(J_{\vec{r}_0}F\big)\big(\vec{r}\,'(1)\big)\text{]} \\ &= \begin{pmatrix} 12 & -2 & 1 \\ 4 & e & 4 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 46 \\ 16 + e \end{pmatrix}.\end{aligned}$$

4. *(15 pts.)* Producing a yacht requires 1 ton of steel and 1 ton of aluminum. Producing an airplane requires 3 tons of steel and 2 tons of aluminum. Producing a ton of steel consumes 1 ton of coal and 2 tons of hematite. Producing a ton of aluminum consumes 4 tons of coal and 2 tons of bauxite. Organize these facts into matrices, and find the matrix that tells you how much coal (c), hematite (h), and bauxite (b) is needed to make y yachts and p airplanes.

Translate the given information into equations:

$$\begin{pmatrix} s \\ a \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} y \\ p \end{pmatrix}, \qquad \begin{pmatrix} c \\ h \\ b \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} s \\ a \end{pmatrix}.$$

Give the matrices names:

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 4 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} c \\ h \\ b \end{pmatrix} = BA\begin{pmatrix} y \\ p \end{pmatrix}$$

where

$$BA = \begin{pmatrix} 5 & 11 \\ 2 & 6 \\ 2 & 4 \end{pmatrix}.$$

5. *(10 pts.)* The *commutator* of two matrices is defined as $[A, B] = AB - BA$. The *trace* of a matrix is the sum of its diagonal elements:

$$\operatorname{tr} M = \sum_j M_{jj}\,.$$

(a) What condition must the matrices A and B satisfy in order for their commutator to be defined?

They must be *square* matrices of the *same size* ($n \times n$).

(b) Prove that the trace of any commutator is equal to zero.

$$\operatorname{tr}(AB) = \sum_{j=1}^{n}(AB)_{jj} = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{jk}B_{kj}$$

by definition of matrix multiplication. Therefore, by interchanging of the matrices, then renaming of indices, then commuting the multiplication of numbers,

$$\operatorname{tr}(BA) == \sum_{j=1}^{n}\sum_{k=1}^{n} B_{jk}A_{kj} = \sum_{k=1}^{n}\sum_{j=1}^{n} B_{kj}A_{jk} = \operatorname{tr}(AB).$$

Thus

$$\operatorname{tr}(AB - BA) = 0,$$

since tr is obviously a linear function of M.

6. *(20 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix}$.

$$\left(\begin{array}{cccccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(1)\leftrightarrow(2)} \left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 2 & 4 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[(3)\to(3)-2(1)]{(2)\to(2)-3(1)}$$

$$\left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -3 & 0 \\ 0 & 4 & 0 & 0 & -2 & 1 \end{array}\right) \xrightarrow{(3)\to(3)-4(2)} \left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 & -3 & 0 \\ 0 & 0 & 8 & -4 & 10 & 1 \end{array}\right) \xrightarrow[(3)\to\frac{1}{8}(3)]{(2)\to(2)+\frac{1}{4}(3)}$$

$$\left(\begin{array}{cccccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{4} & \frac{1}{8} \end{array}\right) \xrightarrow{(1)\to(1)-(3)} \left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{4} & \frac{1}{8} \end{array}\right).$$

Therefore,

$$M^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} \\ 0 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{5}{4} & \frac{1}{8} \end{pmatrix}.$$

Check:

$$MM^{-1} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} \\ 0 & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & \frac{5}{4} & \frac{1}{8} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$