Math. 311 (Fulling) 11 May 1999

Final Examination – Solutions

1. *(25 pts.)* Let $N = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$. Its eigenvalues are 4 and -2 . (You may take these as given!)

(a) Find an **orthonormal** basis of eigenvectors of N .

$\lambda = 4:$ $\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, giving the equations $x-z=0$, $y-z=0$, whose solutions are $x=y=z$. A normalized eigenvector is $\hat{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\lambda = -2:$ $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$ reduces to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, giving the equation $x+y+z=0$, with two standard solutions $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. These are automatically orthogonal to $\hat{u}_1$, but not to each other. So, we apply Gram–Schmidt: Take $\hat{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Then the perpendicular part of $\vec{v}_3$ is

$$\vec{v}_3 - \langle \hat{u}_2, \vec{v}_3 \rangle \hat{u}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}(1+0+0)\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}.$$

So the remaining normalized eigenvector is

$$\hat{u}_3 = \sqrt{\frac{2}{3}}\begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

(b) Suppose that N is the matrix of second derivatives (Hessian) of a function $f(x,y,z)$ at a point where $\nabla f = 0$. Does this point mark a maximum of f , a minimum, or a saddle point?

A saddle point, since the eigenvalues have different signs.

2. *(25 pts.)* Three quantities A , B , C are related by the equations $\begin{cases} AB + BC &= 18, \\ A + B &= 2C. \end{cases}$

(a) Regard A and B as functions of C . Find the derivatives of A and B with respect to C at the point where $A = B = C = 3$.

Implicit differentiation with respect to C yields

$$\begin{aligned} \frac{dA}{dC}\,B + A\,\frac{dB}{dC} + \frac{dB}{dC}\,C + B &= 0, \\ \frac{dA}{dC} + \frac{dB}{dC} &= 2. \end{aligned}$$

Setting all variables equal to 3 (which, we note in passing, does satisfy the original equations!), we get

$$\begin{aligned} 3\frac{dA}{dC} + 6\,\frac{dB}{dC} &= -3, \\ \frac{dA}{dC} + \frac{dB}{dC} &= 2. \end{aligned}$$

The solution (by elementary algebra or by inverting the coefficient matrix) is

$$\begin{pmatrix} \frac{dA}{dC} \\ \frac{dB}{dC} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{1}{3} & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

(b) Find the derivatives of functions $f(A,B)$ and $g(A,B)$ with respect to C at that point, if

$$\begin{pmatrix} \frac{\partial f}{\partial A} & \frac{\partial f}{\partial B} \\ \frac{\partial g}{\partial A} & \frac{\partial g}{\partial B} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$$

there.

Chain rule:

$$\begin{pmatrix} \frac{df}{dC} \\ \frac{dg}{dC} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial A} & \frac{\partial f}{\partial B} \\ \frac{\partial g}{\partial A} & \frac{\partial g}{\partial B} \end{pmatrix} \begin{pmatrix} \frac{dA}{dC} \\ \frac{dB}{dC} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

3. *(30 pts.)* The linear function $L\colon \mathbf{R}^2 \to \mathbf{R}^2$ is represented by the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$ (with respect to the natural basis).

(a) Is L injective (1-to-1)? If not, what is its kernel?

No. The kernel consists of the solutions of $L(\vec{x}) = \vec{0}$, or $x + 2y = 0$. These are the multiples of the vector $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

(b) Is L surjective (onto)? If not, what is its range?

No. The range is the span of the columns, hence the multiples of the vector $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(c) What matrix represents L with respect to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ (used for both domain and codomain)?

The matrix $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ maps coordinates from the new basis to the natural basis. Its inverse is $U^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The matrix we want is

$$U^{-1}AU = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}.$$

4. *(25 pts.)* Let $M = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$.

(a) Find all eigenvalues and eigenvectors of M.

The characteristic equation is

$$0 = \begin{vmatrix} 5-\lambda & 1 \\ 2 & 4-\lambda \end{vmatrix} = (5-\lambda)(4-\lambda) - 2 = \lambda^2 - 9\lambda + 18 = (\lambda - 6)(\lambda - 3).$$

So the eigenvalues are 6 and 3.

$\lambda = 6:$ $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow x = y$. So the eigenvectors are multiples of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = 3:$ $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \Rightarrow x = -\frac{1}{2}y$. So the eigenvectors are multiples of $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

(b) Find the general solution of the system $\begin{cases} \frac{dx}{dt} = 5x + y, \\ \frac{dy}{dt} = 2x + 4y. \end{cases}$

Let $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$. The system becomes the vectorial differential equation $\frac{d\vec{x}}{dt} = M\vec{x}$.

Method 1: The solution is simply

$$\vec{x}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{3t},$$

where c_1 and c_2 are arbitrary constants.

Method 2: Start from the diagonalized form of M : $D = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$. The matrix $U = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ maps coordinates from the eigenbasis to the natural basis. Then $Ue^{tD}U^{-1}$ maps the initial values $\vec{x}(0)$ into $\vec{x}(t)$. For the problem as stated, we don't care about initial values, so we don't really need to calculate U^{-1} . We can just say (with $\vec{c} = U^{-1}\vec{x}(0)$, whatever it is)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} e^{6t}c_1 - e^{3t}c_2 \\ e^{6t}c_1 + 2e^{3t}c_2 \end{pmatrix}.$$

This is the same as the result of Method 1, written out in components.

5. *(20 pts.)* Evaluate the surface integral $\iint_S \vec{B} \cdot d\vec{S}$ if $\vec{B}(\vec{x}) = y\,\hat{\imath} + z\,\hat{k}$ and S is the part of the graph of $z = x^2 + y$ above the region $0 < x < 1$, $0 < y < 2$.

$$\iint_S \vec{B} \cdot d\vec{S} = \iint_S [B_x\, dy\, dz + B_y\, dz\, dx + B_z\, dx\, dy] = \iint_S (y\, dy\, dz + z\, dx\, dy).$$

With $dz = 2x\, dx + dy$ and $dy\, dx = -dx\, dy$, $dy\, dy = 0$, we get

$$\iint_x (-2xy + x^2 + y)\, dx\, dy = \int_0^2 dy \int_0^1 dx\, (-2xy + x^2 + y) = \int_0^2 [-x^2 y + \frac{1}{3}x^3 + xy]_0^1\, dy$$

$$= \int_0^2 \left(\frac{1}{3}\right) dy = \frac{2}{3}.$$

(Note that once you reduce the integral to an ordinary double integral with differentials $dx\, dy$, you can evaluate the iterated integral in either order without changing the sign again!)

6. *(20 pts.)* Do **ONE** of these [(A) or (B)].

   (A) Write out the definition of an **inner product** for a (real) vector space $\mathcal{V}$, and then give an example of an inner product for the function space $\mathcal{V} = \mathcal{C}^2(0,1)$.

See Sec. 6.1 for the definition. The most obvious example is

$$\langle f, g \rangle \equiv \int_0^1 f(t)g(t)\, dt,$$

but there are many others — all basically involving an integral over the interval $(0, 1)$.

(B) Calculate this determinant, **showing all steps**.

Calculators may be used for elementary arithmetic only!

$$\begin{vmatrix} 0 & 0 & 2 & 0 & 0 \\ 1 & -1 & 39 & 1 & -1 \\ 2 & 3 & -7 & 4 & 5 \\ 9 & 2 & 17 & 7 & 6 \\ 1 & 2 & -22 & 1 & 5 \end{vmatrix}$$

Let's do a cofactor expansion, then a row reduction:

$$2\begin{vmatrix} 1 & -1 & 1 & -1 \\ 2 & 3 & 4 & 5 \\ 9 & 2 & 7 & 6 \\ 1 & 2 & 1 & 5 \end{vmatrix} = 2\begin{vmatrix} 1 & -1 & 1 & -1 \\ 0 & 5 & 2 & 7 \\ 0 & 11 & -2 & 15 \\ 0 & 3 & 0 & 6 \end{vmatrix},$$

then two more cofactor expansions:

$$2\begin{vmatrix} 5 & 2 & 7 \\ 11 & -2 & 15 \\ 3 & 0 & 6 \end{vmatrix} = 2\cdot 3\begin{vmatrix} 2 & 7 \\ -2 & 15 \end{vmatrix} + 2\cdot 6\begin{vmatrix} 5 & 2 \\ 11 & -2 \end{vmatrix} = 12\cdot(15+7) + 24\cdot(-5-11) = -120.$$

(At the last step I factored a 2 out of one column of each of the 2×2 determinants before evaluating them.) Of course, there are other correct routes to the answer.

7. *(20 pts.)* Do **ONE** of these [(C) or (D)].

(C) Let x_1 and x_2 be two real numbers. Find a quadrature (numerical integration) formula of the form $\int_0^1 f(t)\,dt = c_1 f(x_1) + c_2 f(x_2)$ by requiring that the formula give the right answer for $f(t) = 1$ and for $f(t) = t$. Then verify that your answer reduces to the usual trapezoidal rule when $x_1 = 0$ and $x_2 = 1$.

The assumed formula, applied to the first two powers, yields

$$1 = \int_0^1 1\,dt = c_1 + c_2\,,$$
$$\frac{1}{2} = \int_0^1 t\,dt = c_1 x_1 + c_2 x_2\,.$$

Solve the equations by your favorite method, getting

$$c_1 = \frac{x_2 - \frac{1}{2}}{x_2 - x_1}\,, \qquad c_2 = \frac{\frac{1}{2} - x_1}{x_2 - x_1}\,.$$

When $x_1 = 0$ and $x_2 = 1$, this becomes $c_1 = c_2 = \frac{1}{2}$. So in that case, the formula is $\int_0^1 f(t)\,dt = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2)$, which is the trapezoidal rule.

(D) Let $\vec{F}(\vec{x}) = z\,\hat{\imath} + \hat{\jmath} + x\,\hat{k}$. (a) Show that a line integral $\int_{\vec{x}_\mathrm{i}}^{\vec{x}_\mathrm{f}} \vec{F}(\vec{x}) \cdot d\vec{x}$ is always independent of the path joining the initial and final points. (b) Find a potential energy function V such that $\vec{F} = -\nabla V$. (c) Explain why parts (a) and (b) are related.

The curl of this vector field is everywhere zero:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z & 1 & x \end{vmatrix} = (1-1)\hat{\jmath} = \vec{0}.$$

This implies that the integrals are independent of path. By the fundamental theorem of calculus, the integral equals $-V(\vec{x}_\mathrm{f}) + V(\vec{x}_\mathrm{i})$. Clearly this is possible only if the integral is independent of path, so exhibiting the potential function is an alternative way of proving that. Conversely, if the integral is independent of path, we can integrate along any convenient path from any fixed starting point (say $\vec{0}$) to $\vec{x}$ to define $-V(\vec{x})$. Alternatively, you can solve the three equations

$$\frac{\partial V}{\partial x} = -z, \quad \frac{\partial V}{\partial y} = -1, \quad \frac{\partial V}{\partial z} = -x$$

and fix the constants of integration by requiring that the three results for V be consistent. I will not type the details of either calculation, but will state that the correct answer is

$$V(x,y,z) = -xz - y,$$

and that $V = -2xz - y$ is **wrong** — see Exercise 3.5.8 on this point!

8. *(20 pts.)* Write an essay on ONE of these topics.

(E) Hyperbolic functions (sinh and cosh) and their applications (for example, to linear differential equations and to analysis of the "twin paradox" in relativity).

See Sec. 4.1, and any textbooks on calculus and differential equations. In terms of initial data, the most natural basis for the solutions of $y'' - k^2 y = 0$ is

$$y_1 = \cosh(kt), \quad y_2 = \frac{1}{k}\sinh(kt);$$

these functions satisfy

$$\begin{aligned} y_1(0) &= 1, & y_2(0) &= 0, \\ y_1'(0) &= 0, & y_2'(0) &= 1, \end{aligned}$$

and therefore the solution with $y(0) = A$, $y'(0) = B$ is $y = Ay_1 + By_2$.

The application to special relativity was discussed in a special lecture this semester. The main idea is that the time $2t$ elapsed on a clock moving along a hyperbolic space-time path is the arc length of the hyperbolic segment, while the time elapsed on a stationary clock is the length of the straight line joining the endpoints of the segment, which is $2\sinh t$. The latter is larger. (More precisely, the "arc length" involved here is not the Euclidean length, but the relativistic proper time defined by the differential relation $ds = \sqrt{dt^2 - dx^2}$.)

(F) Interesting facts about the vector field $\vec{F} = -\frac{y}{x^2+y^2}\,\hat{\imath} + \frac{x}{x^2+y^2}\,\hat{\jmath}$.

[See Sec. 7.5 and Exercise 7.5.8.]

(G) Four-dimensional analogues of the vector cross product.

[See Sec. 7.2.]

(H) The distinction between position vectors and displacement vectors, and their analogy with pointers and array indices in programming languages such as `C`.

[See Sec. 1.3.]

(I) Superposition principles, and how they might be applied in solving Poisson's equation ($\nabla^2 u = \rho$) or boundary-value problems for Laplace's equation ($\nabla^2 u = 0$).

[See Sec. 5.3 and Exercises 5.3.3 and 5.3.4.]

(J) Why the identity $\nabla \times (\nabla f) = 0$ is necessary to prevent an inconsistency between Stokes's theorem and the fundamental theorem of calculus. Hint: Consider two curves that have the same endpoints.

[See Exercise 7.5.3. Also see Questions (D) and (F).]

(K) Prove (from the definitions) that the **range** of a linear function is a **subspace** of the codomain.

[See Sec. 5.2, Theorem 1.]

(L) Prove that for a linear function, the dimension of the range plus the dimension of the kernel equals the dimension of the domain. Hint: A good first sentence is "Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for the kernel."

[See Sec. 5.4, Theorem 1.]

9. **Take 15 free points in honor of your classmates who will graduate this week!**

10. *(20 extra credit pts.)* Do **ONE** more of the optional problems [(A) through (L)].