Math. 311 (Fulling) 17 February 1999

# Test A – Solutions

**Calculators may be used for simple arithmetic operations only!**

1. *(15 pts.)* Find a parametric representation of the plane (in $\mathbf{R}^3$ with coordinates $(x, y, z)$ ) whose equation is $3x - 2z = -2$ .

Let's do this by the formal row-reduction method of Chapter 2. The augmented matrix and its reduced form are

$$\begin{pmatrix} 3 & 0 & -2 & -2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 0 & -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}.$$

Working from the end of the variable list back to the beginning, we see that we must take

$$z = t, \quad y = s \quad \text{(arbitrary parameters)},$$

and then

$$x = \frac{2}{3}\,t - \frac{2}{3}\,.$$

In vector form,

$$\vec{x} = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} \frac{2}{3} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -\frac{2}{3} \\ 0 \\ 0 \end{pmatrix}.$$

It should be noted that there are other correct answers, such as

$$\vec{x} = s\begin{pmatrix} 1 \\ 0 \\ \frac{3}{2} \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Also, there are other ways of presenting an answer, such as

$$\vec{x} = \begin{pmatrix} \frac{2}{3}(t-1) \\ s \\ t \end{pmatrix}.$$

*Remarks:* The answer to a problem like this is easily checked: The constant term must by itself satisfy the equation, since it corresponds to $s = t = 0$ . The vectors multiplied by $s$ and $t$ must satisfy the *corresponding homogeneous* equation, $3x - 2z = 0$ . Unfortunately, passing these tests does not guarantee that your solution is complete: The main lesson of this problem is the necessity of including the term proportional to $(0, 1, 0)$ (or saying something about what the coordinate $y$ does!). If you leave out that term, you have constructed just a line, not a parametrized plane.

2. *(15 pts.)* An A2X30 module contains 1000 cubic inches of steel and 20 cubic inches of titanium. A B62W subassembly contains 10 cubic inches of steel and 1 cubic inch of titanium. A supertanker is built from 10 A2X30s and 8 B62Ws. A minesweeper is built from 5 A2X30s and 3 B62Ws.

   Organize these facts into matrices, and find the matrix that should be used to calculate the quantities of metals needed to make $k$ tankers and $m$ minesweepers.

Let $A$ , $B$ , $s$ . $t$ have the obvious meanings. Translating the sentences into equations, we have

$$\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} 1000 & 10 \\ 20 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \qquad \text{(let's call this matrix } M \text{ ),}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 10 & 5 \\ 8 & 3 \end{pmatrix} \begin{pmatrix} k \\ m \end{pmatrix} \qquad \text{(let's call this matrix } N \text{ ).}$$

(Be sure that your matrices express the correct relationships, calculating the required inputs from the desired outputs. For example, the top line of the first matrix equation expresses the formula $s = 1000A + 10B$ , saying that we need 1000 units of steel for each A module and 10 units of steel for each B module.) Therefore,

$$\begin{pmatrix} s \\ t \end{pmatrix} = MN \begin{pmatrix} k \\ m \end{pmatrix},$$

where

$$MN = \begin{pmatrix} 1000 & 10 \\ 20 & 1 \end{pmatrix} \begin{pmatrix} 10 & 5 \\ 8 & 3 \end{pmatrix} = \begin{pmatrix} 10080 & 5030 \\ 208 & 103 \end{pmatrix}.$$

3. *(20 pts.)* Atoms near the point $\vec{x}_0 \equiv \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ sit in an electric field $\vec{E} = \begin{pmatrix} x^2 - y \\ x^2 + y^2 \\ z^3 \end{pmatrix}$ .

   (a) Find the first-order (best affine) approximation to $\vec{E}(\vec{x})$ for $\vec{x}$ near $\vec{x}_0$ .

   The Jacobian matrix is

$$\frac{d\vec{E}}{d\vec{x}} = \begin{pmatrix} 2x & -1 & 0 \\ 2x & 2y & 0 \\ 0 & 0 & 3z^2 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ at } \vec{x}_0 .$$

Therefore,

$$\vec{E}(\vec{x}) \approx \vec{E}(\vec{x}_0) + \frac{d\vec{E}}{d\vec{x}} (\vec{x} - \vec{x}_0) = \begin{pmatrix} 7 \\ 13 \\ 1 \end{pmatrix} + \begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x-3 \\ y-2 \\ z-1 \end{pmatrix}.$$

(b) Suppose that the index of refraction of a crystal depends on the electric field according to the law

$$n = 1 + 0.01{E_x}^2 + 0.04{E_y}^2 + 0.02{E_z}^2.$$

Use the multidimensional chain rule to find $\dfrac{\partial n}{\partial y}$ at $\vec{x}_0$ .

*Method 1:* $\nabla n = \dfrac{dn}{d\vec{E}}\dfrac{d\vec{E}}{d\vec{x}} = (0.02E_x, 0.08E_y, 0.04E_z)\Big|_{\vec{x}_0}\dfrac{d\vec{E}}{d\vec{x}} =$

$$(0.14, 1.04, 0.04)\begin{pmatrix} 6 & -1 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 3 \end{pmatrix} = (\,*\,, 4.02,\ *\,),$$

where the numbers $*$ are irrelevant and $4.02 = \dfrac{\partial n}{\partial y}$ .

*Method 2:* (This is really the same method, but in "classical" partial-derivative notation instead of vectors and matrices.)

$$\frac{\partial n}{\partial y} = \frac{\partial n}{\partial E_x}\frac{\partial E_x}{\partial y} + \frac{\partial n}{\partial E_y}\frac{\partial E_y}{\partial y} + \frac{\partial n}{\partial E_z}\frac{\partial E_z}{\partial y} = 0.02E_x\,(-1) + 0.08E_y\,(2y) + 0.04E_z\,(0).$$

When all this is evaluated at $(x, y, z) = (3, 2, 1)$ , we again get 4.02.

4. *(15 pts.)* Let's define a mapping $Q$ of the function space $\mathcal{C}^1(0,\infty)$ into the function space $\mathcal{C}(0,\infty)$ by

$$Q(f)(t) \equiv \frac{df}{dt} + (2t+1)f(t)^2.$$

(Here $t$ is the independent variable of the functions in $\mathcal{C}^1(0,\infty)$ , and $f$ is a generic element of $\mathcal{C}^1(0,\infty)$ .) Is $Q$ linear, affine, or nonlinear? Justify your answer.

Nonlinear. It is enough to show that either of the linearity conditions is violated; for instance,

$$Q(2f) = 2\,\frac{df}{dt} + 4(2t+1)f^2 \neq 2\,\frac{df}{dt} + 2(2t+1)f^2 = 2Q(f).$$

5. *(15 pts.)*

(a) Solve the system $\begin{Bmatrix} x + y = 2, \\ x - by = 0 \end{Bmatrix}$ for $x$ and $y$ (with $b$ as a parameter).

Set up an augmented matrix and reduce:

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & -b & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -b-1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 - \frac{2}{b+1} \\ 0 & 1 & \frac{2}{b+1} \end{pmatrix}.$$

We note that the last step assumes $b + 1 \neq 0$ . Therefore, if $b \neq -1$ , we have

$$x = 2 - \frac{2}{b+1} = \frac{2b}{b+1}\,, \qquad y = \frac{2}{b+1}\,.$$

(b) Point out any values of $b$ that are "special" with regard to existence and uniqueness of solutions. (Explain.)

If $b = -1$ , the equations are inconsistent; no solutions exist.

6. *(20 pts.)* Find the inverse (if it exists) of the matrix $M = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & -2 \\ 2 & 1 & 0 \end{pmatrix}$ .

$$\begin{pmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{array}{c} (1) \leftrightarrow (2) \\ (2) \leftarrow (2) - 3(1) \\ (3) \leftarrow (3) - 2(1) \\ \longrightarrow \end{array}$$

$$\begin{pmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -1 & 7 & 1 & -3 & 0 \\ 0 & -1 & 4 & 0 & -2 & 1 \end{pmatrix} \quad \begin{array}{c} (2) \leftarrow -(2) \\ (1) \leftarrow (1) - (2) \\ (3) \leftarrow (3) + (2) \\ \longrightarrow \end{array}$$

$$\begin{pmatrix} 1 & 0 & 5 & 1 & -2 & 0 \\ 0 & 1 & -7 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & -1 & -1 \end{pmatrix} \quad \begin{array}{c} (3) \leftarrow \frac{1}{3}(3) \\ (1) \leftarrow (1) - 5(3) \\ (2) \leftarrow (2) + 7(3) \\ \longrightarrow \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} & -\frac{1}{3} & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} & \frac{2}{3} & -\frac{7}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}.$$

Therefore,

$$M^{-1} = \frac{1}{3} \begin{pmatrix} -2 & -1 & 5 \\ 4 & 2 & -7 \\ 1 & -1 & -1 \end{pmatrix}.$$