Math. 311 (Fulling) 24 March 1999

Test B – Solutions

1. *(18 pts.)* Determine whether each set is linearly independent. If it is not, find an independent set with the same span.

(a) $$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \right\}$$

YES — obviously these two vectors are not multiples of each other. More formally,

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{9}{5} \\ 0 & 1 & -\frac{2}{5} \end{pmatrix},$$

and we see that the number of nonzero rows has not decreased as we put the matrix into row echelon form.

(b) $$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix} \right\}$$

Let's put the vectors into a matrix as rows and reduce:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 1 & 7 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -2 \\ 0 & 5 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{11}{5} \\ 0 & 1 & \frac{2}{5} \\ 0 & 0 & 0 \end{pmatrix}.$$

So the answer is NO. A basis for the span is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{2}{5} \end{pmatrix} \right\}, \qquad \text{or} \qquad \left\{ \begin{pmatrix} 1 \\ 0 \\ \frac{11}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{2}{5} \end{pmatrix} \right\},$$

or any two of the original three vectors (since *in this case* none of the vectors is a multiple of any of the others).

2. *(12 pts.)* Two bases for the solution space of $y'' = 9y$ are

$$\mathcal{A} = \{\vec{a}_1 = e^{3t},\ \vec{a}_2 = e^{-3t}\} \qquad \text{and} \qquad \mathcal{B} = \{\vec{b}_1 = \cosh(3t),\ \vec{b}_2 = \tfrac{1}{3}\sinh(3t)\}.$$

Find the matrix that expresses the **coordinates** of an arbitrary vector with respect to the $\mathcal{A}$ basis in terms of its coordinates with respect to the $\mathcal{B}$ basis.

From the definition of the hyperbolic functions, we have

$$\vec{b}_1 = \tfrac{1}{2}\vec{a}_1 + \tfrac{1}{2}\vec{a}_2\,,$$
$$\vec{b}_2 = \tfrac{1}{6}\vec{a}_1 - \tfrac{1}{6}\vec{a}_2\,.$$

From here there are many equivalent ways to proceed:

Method 1: Therefore,

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{6} \end{pmatrix}^{\mathrm{t}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}$$

maps the $\mathcal{B}$ -coordinates into the $\mathcal{A}$ -coordinates, as demanded.

Method 2: Therefore,

$$(\vec{b}_1, \vec{b}_2) = (\vec{a}_1, \vec{a}_2) \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix},$$

and the matrix appearing in this equation is the desired one.

Method 3: Therefore, if $y = x_1\vec{a}_1 + x_2\vec{a}_2$ and also $y = y_1\vec{b}_1 + y_2\vec{b}_2$, then

$$y = y_1\left(\tfrac{1}{2}\vec{a}_1 + \tfrac{1}{2}\vec{a}_2\right) + y_2\left(\tfrac{1}{6}\vec{a}_1 - \tfrac{1}{6}\vec{a}_2\right) = \left(\tfrac{1}{2}y_1 + \tfrac{1}{6}y_2\right)\vec{a}_1 + \left(\tfrac{1}{2}y_1 - \tfrac{1}{6}y_2\right)\vec{a}_2\,.$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Remark: The factor $\frac{1}{3}$ was not just thrown in to make the problem harder. $\vec{b}_2$ is a "natural" solution to use, because it satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$.

3. *(32 pts.)* A linear mapping $L\colon \mathcal{P}_2 \to \mathcal{P}_2$ is defined by

$$[L(p)](t) = (t^2 - 4)p''(t) + tp'(t) - 4p(t) \qquad \left(p' \equiv \frac{dp}{dt}\,, \text{etc.}\right).$$

Note that $L(t^2) = -8$ (more precisely: if $p(t) = t^2$, then $L(p)(t) = -8$ for all t). (This is free information! You don't have to rederive it.)

(a) Find the matrix that represents L with respect to the standard basis $\{t^2, t, 1\}$ for $\mathcal{P}_2$.

$$\begin{aligned} L(t^2) &= -8, \\ L(t) &= 0 + t - 4t = -3t, \\ L(1) &= -4. \end{aligned} \qquad \text{Therefore, the matrix is} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ -8 & 0 & -4 \end{pmatrix}.$$

(b) Is L surjective (onto $\mathcal{P}_2$)? If not, what is its range?

NO. The range is $\mathcal{P}_1$, the first-degree polynomials. This is clear either from the matrix — the span of the columns being the same as the span of

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

— or from the action of L on the basis polynomials, which yields the constants and the multiples of t but no t^2 terms.

(c) Is L injective? If not, what is its kernel?

NO. From (b) and the dimension theorem, the kernel must have dimension $3 - 2 = 1$. To see what the kernel is, reduce the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ -8 & 0 & -4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus $p(t) = at^2 + bt + c$ is in the kernel if

$$a + \tfrac{1}{2}c = 0, \qquad b = 0.$$

That is, the kernel consists of the multiples of the polynomial

$$-\frac{1}{2}t^2 + 1.$$

(d) Use “superposition principles” to find **all** solutions in $\mathcal{P}_2$ of the differential equation

$$(t^2 - 4)p''(t) + tp'(t) - 4p(t) = -16.$$

Obviously one solution is $p_{\mathrm{p}}(t) = 2t^2$. The general solution, therefore, is $p = p_{\mathrm{p}} + p_{\mathrm{c}}$, where p_{c} is the general element of the kernel (general solution of the corresponding homogeneous equation, $L(p) = 0$). Thus, all the solutions in $\mathcal{P}_2$ are

$$p(t) = 2t^2 + \left(-\tfrac{1}{2}t^2 + 1\right)c \quad \text{for arbitrary numbers } c\,.$$

(Of course, there are other solutions of the differential equation that are not quadratic polynomials.)

4. *(22 pts.)* The matrix $M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix}$ represents a linear operator $L\colon \mathbf{R}^3 \to \mathbf{R}^2$ with respect to the natural bases.

(a) What are the rank and the nullity of M ? (The nullity is the dimension of the kernel of L .)

The rank is 2 : *Method 1:* It can't be greater than 2, because the dimension of the whole column space is just 2. It can't be less than 2, because the three columns are not all proportional. *Method 2:* The (column) rank is equal to the row rank, and we already saw in Qu. 1(a) that the row rank of this matrix is 2 .

Therefore, by the dimension theorem, the nullity is $3 - 2 = 1$. This can also be checked directly, by reducing the augmented matrix to (compare Qu. 1(a))

$$\begin{pmatrix} 1 & 0 & \frac{9}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \end{pmatrix},$$

from which it is clear that the third coordinate of the solution is arbitrary and the first and second coordinates are then determined.

(b) What matrix represents L if we switch to the basis $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ for $\mathbf{R}^2$? (The basis for the domain is still the natural one.)

Call the new basis vectors $\vec{b}_1$ and $\vec{b}_2$. In terms of the natural basis vectors, we have

$$\vec{b}_1 = \hat{e}_1 + \hat{e}_2 \,,$$
$$\vec{b}_2 = -\hat{e}_1 + \hat{e}_2 \,.$$

Therefore (reasoning as in Qu. 2),

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

maps $\vec{b}$ -coordinates into natural coordinates. (More quickly, this matrix is obtained simply by "stacking the new basis vectors together".) Therefore,

$$C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

maps natural coordinates to $\vec{b}$ -coordinates. To solve our problem we need to postprocess the natural-basis calculation of L with C^{-1} . Therefore, the desired matrix is

$$C^{-1}M = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 3 & 1 & 5 \\ 1 & -3 & 3 \end{pmatrix}.$$

5. *(16 pts.)*

(a) Explain what a "subspace" is.

A subspace of a vector space $\mathcal{V}$ is a subset of $\mathcal{V}$ that is *closed* under addition and scalar multiplication. The meaning of "closed" is made clear by the proofs below.

(b) Prove that one of these is a subspace (of a vector space $\mathcal{V}$). [Do **either** (A) **or** (B) – your choice.]

(A) The kernel of a linear function $L\colon \mathcal{V} \to \mathcal{V}$.

Suppose that we have two elements of the kernel: $L(\vec{v}_1) = 0 = L(\vec{v}_2)$. Then

$$L(r\vec{v}_1 + \vec{v}_2) = rL(\vec{v}_1) + L(\vec{v}_2).$$

That is, $r\vec{v}_1 + \vec{v}_2$ is also in the kernel. Thus the kernel is closed under the vector operations.

(B) The span of a list of vectors in $\mathcal{V}$, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N\}$.

Suppose that we have two elements of the span:

$$\vec{u}_1 = \sum_{j=1}^{N} c_j \vec{v}_j \quad \text{and} \quad \vec{u}_2 = \sum_{j=1}^{N} d_j \vec{v}_j \,.$$

Then

$$r\vec{u}_1 + \vec{u}_2 = r\sum_{j=1}^{N} c_j \vec{v}_j + \sum_{j=1}^{N} d_j \vec{v}_j = \sum_{j=1}^{N} (rc_j + d_j)\vec{v}_j \,,$$

which is also an element of the span, as was to be proved.