Math. 311 (Fulling) 23 April 1999

Test C – Solutions

(You may want to do the last problem first. I had to put it at the end to get good page breaks.)

1. *(25 pts.)* The formula $\langle p, q\rangle = \int_0^\infty p(t)q(t)\,e^{-t}\,dt$ defines an inner product on the vector space of polynomials. Find the first three of the orthonormal polynomials associated with this inner product. (Apply the Gram–Schmidt algorithm to the power functions.) FREE INFORMATION: $\int_0^\infty t^n\,e^{-t}\,dt = n!$.

   Let $v_0 = t^0$, etc., and let $\hat{u}_0$, etc., be the resulting orthogonal polynomials.
   Step 1: We have $\|v_0\|^2 = \int_0^\infty e^{-t}\,dt = 0! = 1$. Thus

$$\hat{u}_0 = v_0 = 1$$

(that is, the *function* whose value is 1 at every value of t).

Step 2: We find $\langle \hat{u}_0, v_1\rangle = \int_0^\infty t\,e^{-t}\,dt = 1! = 1$, so $v_{1\|} = \langle \hat{u}_0, v_1\rangle \hat{u}_0 = 1$. Therefore, $v_{1\perp} = v_1 - v_{1\|} = t - 1$. Then we calculate

$$\|v_{1\perp}\|^2 = \int_0^\infty (t^2 - 2t + 1)\,e^{-t}\,dt = 2! - 2 + 1 = 1,$$

and conclude

$$\hat{u}_1 = t - 1.$$

Step 3:

$$\langle \hat{u}_0, v_2\rangle = \int_0^\infty t^2 e^{-t}\,dt = 2! = 2, \qquad \langle \hat{u}_1, v_2\rangle = \int_0^\infty (t-1)t^2 e^{-t}\,dt = 3! - 2! = 4,$$

so

$$v_{2\|} = \langle \hat{u}_0, v_2\rangle \hat{u}_0 + \langle \hat{u}_1, v_2\rangle \hat{u}_1 = 2 + 4(t-1) = 4t - 2,$$

and hence $v_{2\perp} = t^2 - 4t + 2$. Then

$$\begin{aligned}
\|v_{2\perp}\|^2 &= \int_0^\infty (t^2 - 4t + 2)^2 e^{-t}\,dt \\
&= \int_0^\infty (t^4 - 8t^3 + 16t^2 + 4t^2 - 16t + 4)e^{-t}\,dt \\
&= 4! - 8\cdot 3! + 20\cdot 2! - 16 + 4 = 24 - 48 + 40 - 12 = 4.
\end{aligned}$$

Therefore,

$$\hat{u}_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|} = \frac{1}{2}\,t^2 - 2t + 1.$$

2. *(35 pts.)* Let $\vec{F}(\vec{x}) = x\,\hat\imath + y\,\hat\jmath + (x^2+y^2)\,\hat{k}$. Let (r, θ, z) be standard cylindrical coordinates ($r^2 = x^2 + y^2$, etc.).

(a) Calculate $\nabla \cdot \vec{F}$ (the divergence).

$$\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x^2 + y^2) = 1 + 1 + 0 = 2.$$

(b) Calculate $\nabla \times \vec{F}$ (the curl).

$$\begin{vmatrix} \hat\imath & \hat\jmath & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & x^2 + y^2 \end{vmatrix} = \hat\imath(2y - 0) + \hat\jmath(0 - 2x) + \hat{k}(0 - 0) = 2y\,\hat\imath - 2x\,\hat\jmath.$$

(c) Calculate $\iint_S \vec{F} \cdot d\vec{S}$, where the surface S is the "soup can" consisting of the cylinder

$C:$ $$r = 1, \quad 0 \le \theta < 2\pi, \quad 0 < z < 3$$

and the top and bottom disks

$$0 \le r < 1, \quad 0 \le \theta < 2\pi, \quad z = 0 \text{ or } 3.$$

(The normal vector points outward.)

Since $\nabla \cdot \vec{F} = 2$, by Gauss's theorem this integral is 2 times the volume of the can:

$$2 \cdot \pi r^2 \cdot h = 2 \cdot \pi \cdot 3 = 6\pi.$$

(d) Calculate $\iint_C \vec{F} \cdot d\vec{S}$, where C is the open cylinder described in the previous part (**without** the top and bottom pieces).

Method 1: Since F_z doesn't depend on z , the fluxes through the top and the bottom of the can exactly cancel, so the integral is the same as in (c): 6π .

Method 2: The element of surface area on the cylinder is $dS = r\,d\theta\,dz$, and the normal vector to the cylinder is $\hat{n} = \cos\theta\,\hat\imath + \sin\theta\,\hat\jmath$ (or $\frac{x}{r}\,\hat\imath + \frac{y}{r}\,\hat\jmath$). Therefore,

$$\vec{F} \cdot \hat{n} = x\cos\theta + y\sin\theta = r(\cos^2\theta + \sin^2\theta) = r,$$

so the integral is

$$\iint_C (\vec{F} \cdot \hat{n})\,dS = \int_0^{2\pi} d\theta \int_0^3 dz\,\{r^2\} = 2\pi \cdot 3 \cdot 1 = 6\pi.$$

3. *(23 pts.)* The formulas

$$x = \cosh u \cos v,$$
$$y = \sinh u \sin v$$

define *elliptic coordinates* (u, v) in the $x-y$ plane. The natural ranges of the coordinates are $u \geq 0$, $0 \leq v < 2\pi$. (The curve $u = 0$ is a line segment joining the points at $(-1, 0)$ and $(1, 0)$, which are the foci of the ellipses that give this coordinate system its name. The curves $v = 0$, $\frac{\pi}{2}$, π , and $\frac{3\pi}{2}$ are parts of the x and y axes. All the other coordinate curves are "generic" and are genuinely curved.)

(a) Find the tangent vectors to the coordinate curves $u =$ constant and $v =$ constant (at each point (u, v)).

These vectors are, in the order listed,

$$\frac{\partial \vec{x}}{\partial v} = \begin{pmatrix} -\cosh u \sin v \\ \sinh u \cos v \end{pmatrix}, \qquad \frac{\partial \vec{x}}{\partial u} = \begin{pmatrix} \sinh u \cos v \\ \cosh u \sin v \end{pmatrix}.$$

(Note that at the next step we need to put them in the other order to get the correct Jacobian matrix.)

(b) Find the normal vectors to the coordinate curves. (Introduce appropriate intermediate notation to save writing.)

Let J stand for the determinant of the Jacobian matrix:

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \sinh u \cos v & -\cosh u \sin v \\ \cosh u \sin v & \sinh u \cos v \end{vmatrix}$$
$$= \sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v.$$

(J can be simplified to $\sinh^2 u + \sin^2 v$ or to $\cosh^2 u - \cos^2 v$, but that's not important.) Therefore, the inverse of the Jacobian matrix, by the easy 2×2 Cramer rule, is

$$\frac{1}{J} \begin{pmatrix} \sinh u \cos v & \cosh u \sin v \\ -\cosh u \sin v & \sinh u \cos v \end{pmatrix}.$$

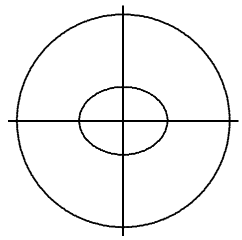
The normal vectors are the rows of this:

$$\nabla u = \frac{1}{J}(\sinh u \cos v,\ \cosh u \sin v),$$
$$\nabla v = \frac{1}{J}(-\cosh u \sin v,\ \sinh u \cos v).$$

(c) Find the area of the region bounded by the curves $u = 1$ and $u = 4$. ("Set up" the integral; you don't need to evaluate it.)

$$\int_0^{2\pi} dv \int_1^4 du \, \{J(u, v)\},$$

where J is as in (b). (The region is the ring between two ellipses — see part (d). I drew the ellipse $u = 2$ because the one with $u = 4$ is huge!)



(d) *(15 points extra credit.* **Graphing calculators are not allowed.)** Sketch one generic coordinate curve of each type. At their intersection point, sketch the two tangent vectors. Describe in words how a sketch of the two normal vectors would differ from the one you drew. FREE INFORMATION: $\cosh^2 u - \sinh^2 u = 1$; $\cosh u$ and $\sinh u$ are positive whenever $u > 0$.

Let's start by finding Cartesian equations for the two kinds of curves. Note that

$$x^2 = \cosh^2 u \cos^2 v, \qquad y^2 = \sinh^2 u \sin^2 v.$$

Combine these equations in such a way as to eliminate v :

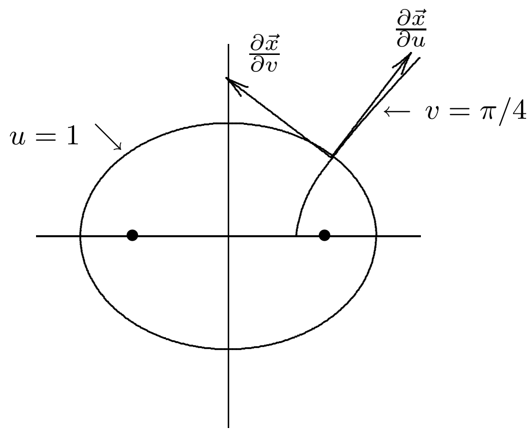
$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1.$$

For fixed u this is the equation of an ellipse (as expected); Since $\cosh u > \sinh u$, the long axis of the ellipse is horizontal. Since we agreed that v would vary over the whole interval of length 2π , the ellipse goes all the way around. (To avoid double-labelling of points, u is restricted to be positive, and this means that each curve of constant v will be only a part of the locus of the related $x{-}y$ equation, as we'll see next.)

Combine the equations so as to eliminate u :

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

This is a hyperbola, cutting the horizontal axis at $x = \pm \cos v$. (Notice that this is always inside the line segment joining the foci, no matter how big u is!) Looking back at the original equation for x , and knowing that $\cosh u > 0$, we see that in fact $x = +\cos v$ (which can be positive or negative, depending on the quadrant of v). Furthermore, since y has the same sign as $\sin v$, the coordinate curve is only half of this branch of the hyperbola (again, either the top or the bottom, depending on v); that is, only a quarter of the complete hyperbolic locus.



The normal vectors in this case are (see (b) and (a)) proportional to the respective tangent vectors (∇u parallel to $\frac{\partial \vec{x}}{\partial u}$, etc.) but of different length. In fact, the lengths are *reciprocal*, so that the inner product of a normal with its respective tangent is 1 . This coordinate system also has the special property that both tangent vectors at a given point have the same length (namely, $\sqrt{J(u,v)}$), in addition to being orthogonal. Such a coordinate system is called *conformal*.

4. *(17 pts.)* Let $\vec{u}_1 = (3,2,1)$, $\vec{u}_2 = (2,0,0)$, $\vec{u}_3 = (5,2,3)$.

(a) Find the volume of the parallelepiped determined by these three vectors.

$$\begin{vmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \\ 5 & 2 & 3 \end{vmatrix} = -2 \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = -2(6-2) = -8.$$

Therefore, the volume is 8 cubic units.

(b) If you set up a coordinate system in physical space with the x , y , and z axes directed along the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ **in that order**, would that system be considered "right-handed" or "left-handed"?

Left-handed, because the determinant is negative.