Math. 312 12 December 2000

Final Examination – Solutions

Some possibly useful information

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Laplacian operator in spherical coordinates:

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi).$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z).$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n(n+1)}{z^2}\right) Z = 0 \quad \text{has solutions } j_n(z) \text{ and } y_n(z).$$

Legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta).$$

Famous Green function integrals:

$$\int_{-\infty}^{\infty} e^{i\omega x}\, e^{-\omega^2 t}\, d\omega = \sqrt{\frac{\pi}{t}}\, e^{-x^2/4t}$$

$$\int_{-\infty}^{\infty} e^{i\omega x}\, e^{-|\omega| y}\, d\omega = \frac{2y}{x^2+y^2}$$

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1. *(8 pts.)* Consider the equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{6}\, u^3$.

(a) Classify it as homogeneous linear, nonhomogeneous linear, or nonlinear.

nonlinear

(b) Classify it as elliptic, hyperbolic, or parabolic.

hyperbolic

Note: In the next three problems, the radius R is a *constant*.

2. *(40 pts.)* Consider the heat equation in a ring:

$$\frac{\partial u}{\partial t} = \frac{1}{R^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{for } 0 \le \theta < 2\pi, \quad 0 < t < \infty,$$

$$\text{periodic boundary conditions in } \theta, \qquad u(0,\theta) = f(\theta).$$

(a) Solve it by separation of variables.

Let $u_{\text{sep}} = T(t)\Theta(\theta)$. Then $T'\Theta = R^{-2}T\Theta''$, so

$$R^2\,\frac{T'}{T} = \frac{\Theta''}{\Theta} = \text{constant} = -n^2$$

where n is an integer because of the periodic boundary conditions. Thus $T = e^{-(n/R)^2 t}$. The general solution is

$$u(t,\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} e^{-(n/R)^2 t}.$$

We must have

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

so

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta.$$

(b) Find a Green function for the problem by the method of images. (Let $R = 1$ in this part, so that $\theta = x$ in our usual notation.)

The Green function for the heat equation on the whole real line is

$$H(t,x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(x-y)} e^{-\omega^2 t}\, d\omega = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}.$$

The Green function for the ring should have image sources spaced a distance 2π apart:

$$G(t,x,y) = \sum_{N=-\infty}^{\infty} H(t,x,y+2\pi N) = \sum_{N=-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y-2\pi N)^2/4t}.$$

(The sign accompanying $2\pi N$ in these formulas is arbitrary.) This sum can't be simplified.

Remark: From part (a) we can get another formula for the Green function:

$$G(t,x,y) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(x-y)} e^{-n^2 t}.$$

This sum can't be simplified either. Its equivalence with the image solution is an instance of the Poisson summation formula.

3. *(40 pts.)* Solve Laplace's equation in a semiinfinite cylinder with boundary data given at the end:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (0 \le r < R, \quad 0 \le \theta < 2\pi, \quad 0 < z < \infty),$$

$$\text{periodic boundary conditions in } \theta, \qquad u(R,\theta,z) = 0,$$

$$u(r,\theta,0) = f(r)\cos\theta, \qquad u \to 0 \text{ as } z \to +\infty.$$

We can shorten the calculations by noting that the Fourier cosine series for the data consists of a single term, so the same will be true for the solution. Consider $u_{\text{sep}} = R(r)Z(z)\cos\theta$:

$$R''Z + \frac{1}{r}R'Z - \frac{1}{r^2}RZ + RZ'' = 0$$

after dividing by $\cos\theta$. Separate the remaining variables:

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} - \frac{1}{r^2} = \text{constant} \equiv -\omega^2 = -\frac{Z''}{Z}.$$

The solution of the Z equation that vanishes at $+\infty$ is $e^{-\omega z}$ (if ω is positive). The radial equation is

$$R'' + \frac{1}{r}R' + \left(\omega^2 - \frac{1}{r^2}\right)R = 0,$$

which is essentially Bessel's equation with $n = 1$. The relevant solution is $J_1(\omega r)$ with ω chosen so that the normal mode vanishes at $r = R$; so define ω_j to be $\frac{1}{R}$ times the jth zero of J_1. Then

$$u(r,\theta,z) = \sum_{j=1}^{\infty} a_j J_1(\omega_j r)\cos\theta\, e^{-\omega_j z}.$$

The coefficient formula is

$$a_j = \frac{\int_0^R J_1(\omega_j r) f(r)\, r\, dr}{\int_0^R J_1(\omega_j r)^2 r\, dr}.$$

4. *(40 pts.)* Solve Laplace's equation in the exterior of a ball,

$$\nabla^2 u = 0 \quad \text{for } R < r < \infty, \qquad u(R,\theta,\phi) = f(\theta).$$

(θ is the *polar* angle, not the azimuthal one. Require u to go to 0 as $r \to \infty$.)

Let $u_{\text{sep}} = R(r) Y_l^m(\theta,\phi)$ and use some of the formulas on the first page of the test to see that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \frac{1}{r^2}\, l(l+1) R.$$

This is not the spherical-Bessel-function equation, because the constant term is missing. Instead, its solutions are r^l and $r^{-(l+1)}$. The second of these is the one that vanishes at infinity. Since the boundary data are independent of ϕ, only spherical harmonics with $m = 0$ contribute. The general solution is

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} a_l r^{-(l+1)} Y_l^0(\theta,\phi),$$

and by orthonormality of the harmonics,

$$\begin{aligned} a_l &= R^{l+1} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \, Y_l^0(\theta,\phi)^* \, f(\theta) \\ &= 2\pi R^{l+1} \int_0^{\pi} Y_l^0(\theta,\phi)^* \, f(\theta) \sin\theta \, d\theta, \end{aligned}$$

where the spherical harmonic is actually independent of ϕ (and real, so the * is unnecessary). Alternatively, we can write the general solution as

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} b_l r^{-(l+1)} P_l(\cos\theta),$$

so that

$$\begin{aligned} b_l &= \frac{R^{l+1} \int_0^{\pi} P_l(\cos\theta) f(\theta) \sin\theta \, d\theta}{\int_0^{\pi} P_l(\cos\theta)^2 \sin\theta \, d\theta} \\ &= \frac{(2l+1)R^{l+1}}{2} \int_0^{\pi} P_l(\cos\theta) f(\theta) \sin\theta \, d\theta. \end{aligned}$$

5. *(40 pts.)* Consider the wave equation on a half-line,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < \infty, \quad -\infty < t < \infty,$$

with boundary condition

$$\frac{\partial u}{\partial x}(t,0) = \alpha u(t,0) \quad (\alpha = \text{positive constant}).$$

(a) Find the *general solution* by separation of variables. (Don't worry about matching initial data. *Do* write a sum or integral of the appropriate form.)

As usual the separated solutions will be $X(x)T(t)$ where $T(t) = \sin(\omega t)$ or $\cos(\omega t)$ and $X(x)$ is some linear combination of $\sin(\omega x)$ and $\cos(\omega x)$, say $X_\omega(x) = \cos(\omega x) + b(\omega)\sin(\omega x)$. The latter must satisfy $X'(0) = \alpha X(0)$, or $\omega b(\omega) = \alpha$, so

$$X_\omega(x) \equiv \cos(\omega x) + \frac{\alpha}{\omega}\sin(\omega x).$$

There is no second boundary condition to constrain ω, so we expect all positive values of ω to contribute. The general solution will be a kind of generalized Fourier transform,

$$u(t,x) = \int_0^\infty A(\omega) X_\omega(x)\cos(\omega t)\,d\omega + \int_0^\infty B(\omega) X_\omega(x)\sin(\omega t)\,d\omega.$$

The coefficient functions A and B will be some integrals (over x) of X_ω times initial data, but it is not immediately obvious how to normalize these integral transforms, which is why you were not required to handle initial data. (It is also legitimate to let $T(t) = e^{i\omega t}$ and let ω run from $-\infty$ to ∞.)

One might worry whether negative or complex eigenvalues might appear in this problem. The complex case can be excluded because the differential operator d^2/dx^2 with the given boundary condition is self-adjoint, and I will not consider that case further. Negative values of the separation constant, say ω^2 replaced by $-\kappa^2$ ($\kappa > 0$), will not appear because α is positive; this we can easily see explicitly. One would have $X(x) = c_1(\kappa)e^{\kappa x} + c_2(\kappa)e^{-\kappa x}$, with

$$\kappa(c_1 - c_2) = \alpha(c_1 + c_2).$$

But c_1 would have to be 0 to have good behavior at $+\infty$, and then the equation for c_2 is inconsistent because κ and α are both positive.

(b) Rearrange your solution to verify that it is consistent with the d'Alembert solution (that is, that it consists of left-moving and right-moving waves).

We need the product formulas for trig functions. If, like me, you can never remember these, there are two ways to proceed: (1) Write out everything in terms of complex exponentials. (2) Use the addition formulas,

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$

and solve for what you need:

$$\begin{aligned}
2\sin a\cos b &= \sin(a+b)+\sin(a-b),\\
2\cos a\sin b &= \sin(a+b)-\sin(a-b),\\
2\cos a\cos b &= \cos(a+b)+\cos(a-b),\\
2\sin a\sin b &= -\cos(a+b)+\cos(a-b).
\end{aligned}$$

Every term in our solution is of one of these forms, with $a=\omega x$ and $b=\omega t$, so the solution is a sum of trig functions of $a+b=\omega(x+t)$ and $a-b=\omega(x-t)$ (integrated over ω). This is enough to keep d'Alembert happy.

6. *(32 pts.)* For each of the "big" problems (Questions 2, 3, 4, 5) —

   (i) Classify the equation as elliptic, parabolic, or hyperbolic in "type".

   (ii) Point out a feature of the solution that is associated with the type of the equation. (The feature need not be obvious from the formula you found for the solution. Try not to use the same feature twice.)

2. parabolic. Features include
- Problem is well-posed in the future direction of time only.
- Solution is smooth (even if the data function isn't).
- Maximum and minimum values of the solution are attained at the initial time.
- Infinite propagation speed ("instant spreadout") for information in the initial data.

3/4. elliptic. Features include
- Well-posed problem requires one data function on each boundary surface (not initial data).
- Solution is smooth (even if the data functions aren't).
- Maximum and minimum values of the solution are attained on the boundary. On the other hand, this is not true for the eigenfunctions $J_1(\omega_j r)\cos\theta$ in Qu. 3, although they satisfy an elliptic equation, because the constant term in that equation is not the one to which the maximum principle applies.

5. hyperbolic. Features include
- Second-order initial-value problem: both the solution and its derivative need to be prescribed as initial data.
- Solution is as rough as the data.
- Finite propagation speed. (Part (b) was a strong hint here.)