Math. 312 22 September 2000

Test A – Solutions

1. *(30 pts.)* Consider the wave equation on an interval,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < 1, \quad -\infty < t < \infty),$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(1,t)$$

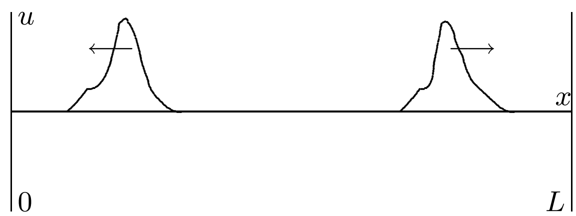
and initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

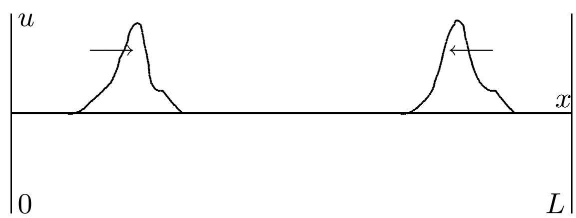
Describe in words and sketches (and possibly a few equations) what the solution is like, assuming that $f(x)$ is a sharply peaked function such as

$$f(x) = e^{-10\left(x-\frac{1}{2}\right)^2}.$$

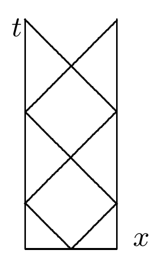
Initially the peak divides into two equal parts, left-moving and right-moving, according to the D'Alembert formula (for the case with zero initial derivative), $\frac{1}{2}[f(x+t) + f(x-t)]$:



Whenever each wave packet hits a boundary, it reflects back. In this case it stays right side up, because the boundary condition is one of vanishing derivative (yielding the *even* periodic extension as the effective initial condition for the equivalent full-space problem). Thus after the first reflection we have



(For simplicity I assume that the packet is originally located at the center of the interval, as in the example given in the problem.) The paths of the bouncing packets in space-time are like this:



2. *(35 pts.)* *(You can answer (b) and (c) without doing (a) first.)*

(a) Find the Fourier cosine series for the function defined by

$$f(x) = e^x \quad \text{on the interval } 0 \le x \le \pi.$$

Hint: $$\int e^{ax} \cos(bx)\, dx = \frac{e^{ax}[a\cos(bx) + b\sin(bx)]}{a^2 + b^2} \quad \text{(from an integral table)}$$

The interval has length $L = \pi$, so $\frac{n\pi}{L} = n$. So the series is

$$f(x) \sim \sum_{n=0}^{\infty} a_n \cos(nx)$$

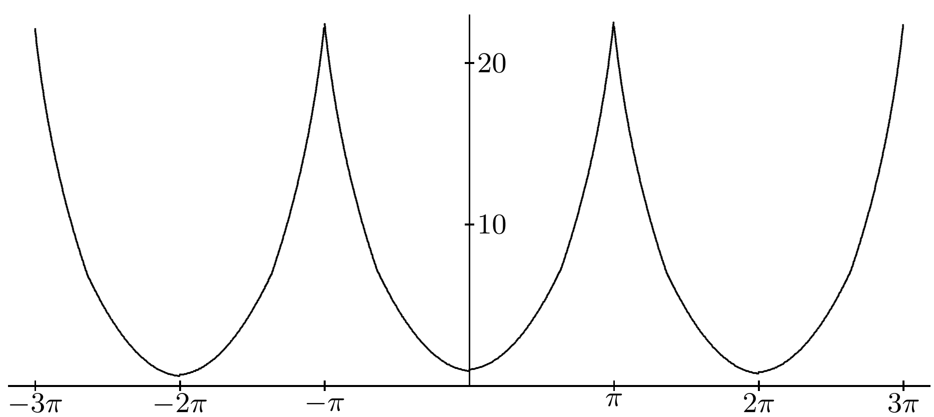
with

$$a_0 = \frac{1}{\pi}\int_0^\pi e^x\, dx = \frac{e^\pi - 1}{\pi},$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi e^x \cos(nx) \\ &= \frac{2}{\pi}\left.\frac{e^x[\cos(nx) + n\sin nx]}{1+n^2}\right|_0^\pi \\ &= \frac{2}{\pi}\frac{e^\pi(-1)^n - 1}{1+n^2}. \end{aligned}$$

(b) This series represents a periodic function defined on the whole line $-\infty < x < \infty$. Sketch the graph of that function over several periods (say from -3π to 3π).

This is the even periodic extension, with period 2π.



(The curve has a (very blunt) corner at each minimum, rather than being smooth like a parabola.)

(c) Is the series uniformly convergent?

Yes, because the periodically extended function is continuous and piecewise smooth.

3. *(35 pts.)* Solve by separation of variables the equation

$$m^2 u + \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \quad -\infty < t < \infty),$$

with boundary conditions

$$u(0,t) = 0 = u(L,t)$$

and initial conditions

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You may assume as common knowledge that all eigenvalues K of the problem

$$X''(x) + KX(x) = 0, \quad X(0) = 0 = X(L)$$

are positive real numbers.

First look for separated solutions $u_{\text{sep}}(x,t) = X(x)T(t)$. Put this into the PDE and divide by XT:

$$m^2 + \frac{T''}{T} = \frac{X''}{X} = \text{constant},$$

since the left side depends only on t and the right side only on x. Call the constant $-\lambda^2$. Because of the boundary conditions, we must have

$$X'' + \lambda^2 X = 0, \quad X(0) = 0 = X(L).$$

The only solutions are (up to constant factors)

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad \lambda_n = \frac{n\pi}{L}, \quad n = 1, 2, \ldots.$$

The t equation is

$$T'' + (m^2 + \lambda^2)T = 0,$$

with solutions

$$T(t) = C\cos\left(t\sqrt{m^2+\lambda^2}\right) + D\sin\left(t\sqrt{m^2+\lambda^2}\right).$$

Since the initial condition $u(x,0) = 0$ is homogeneous, we can impose it already at the separated stage (or wait to impose it later, at the cost of more writing); it implies that $C = 0$. So the general solution of the equation is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\sin\left(t\sqrt{m^2+\lambda_n^2}\right).$$

To compute the coefficients we must use the other initial condition,

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(\lambda_n x)\sqrt{m^2+\lambda_n^2}.$$

Using the Fourier sine coefficient formula for an interval of length L we get

$$b_n = \frac{1}{\sqrt{m^2+\lambda_n^2}}\,\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\, g(x)\, dx.$$