An Example with Time-Dependent Boundary Conditions

(The Mathematical Theory of Vegetable Cellars)

Problem statement

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty,\ 0 < t < \infty),$$

$$u(0,t) = g(t).$$

Let us not state an initial condition yet, because we know that if we find *any* solution of the linear conditions thus far, we can subtract it from an unknown solution and get a problem with *homogeneous* boundary data; the initial data (suitably adjusted) can be handled by that familiar problem.

Method 1: Oscillatory steady state

Reference: H. Dym and H. P. McKean, *Fourier Series and Integrals* (Academic Press, 1972), Sec. 1.8.5; they in turn cite A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press, 1949).

Let's consider the special boundary data

$$u(0,t) = \sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right).$$

Analogy with the usual steady-state solution for a t-independent boundary condition suggests that we might find a solution u that oscillates with the same frequency as the boundary data.

Remark: Another analogy is to the solution of a forced, constant-coefficient ODE by the "method of undetermined coefficients". Such a solution is called (especially in engineering applications) a "steady-state solution" although it does depend on t. The complementary solution of the homogeneous ODE is called the "transient solution", because in most cases it decays exponentially in time. The solution of the homogenized PDE mentioned above is the analogue of the transient for our problem.

Accordingly, we look for a solution

$$u(x,t) = C_+(x)e^{it} + C_-(x)e^{-it}$$

and calculate

$$\frac{\partial u}{\partial t} = i[C_+(x)e^{it} - C_-(x)e^{-it}], \qquad \frac{\partial^2 u}{\partial x^2} = C_+''(x)e^{it} + C_-''(x)e^{-it}.$$

It follows that

$$C_+''(x) = iC_+(x), \quad C_-''(x) = -iC_-(x).$$

On the other hand,

$$\sin t = u(0,t) = C_+(0)e^{it} + C_-(0)e^{-it}$$

implies

$$C_+(0) = \frac{1}{2i}\,, \qquad C_-(0) = -\,\frac{1}{2i}\,.$$

We therefore have

$$C_+(x) = A_+ e^{x\sqrt{i}} + B_+ e^{-x\sqrt{i}} \quad \text{where } A_+ + B_+ = \frac{1}{2i}\,,$$

$$C_-(x) = A_- e^{x\sqrt{-i}} + B_- e^{-x\sqrt{-i}} \quad \text{where } A_- + B_- = -\,\frac{1}{2i}\,,$$

At this point you have to know or figure out that

$$\pm\sqrt{i} = \pm\,\frac{1+i}{\sqrt{2}}\,, \qquad \pm\sqrt{-i} = \pm\,\frac{-1+i}{\sqrt{2}}\,.$$

(Even if you haven't taken complex analysis, you can check that the squares come out right.)

We now make our usual assumption for infinite spatial intervals, that the solution decays as $x \to \infty$; in other words, the real part of the exponents must be negative. That says that

$$A_+ = 0, \quad B_- = 0, \quad B_+ = -\,\frac{i}{2}\,, \quad A_- = \frac{i}{2}\,.$$

$$\begin{aligned} u(x,t) &= -\,\frac{i}{2}\,e^{-x(1+i)/\sqrt{2}}e^{it} + \frac{i}{2}\,e^{-x(1-i)/\sqrt{2}}e^{-it} \\ &= -\,\frac{i}{2}\,e^{-x/\sqrt{2}}\left[e^{it-ix/\sqrt{2}} - e^{-it+ix/\sqrt{2}}\right] \\ &= e^{-x/\sqrt{2}}\sin\left(t - \frac{x}{\sqrt{2}}\right). \end{aligned}$$

That's our answer, which can easily be checked to satisfy the PDE and the boundary condition. Let's interpret it. The temperature distribution $u(x,t)$ is *damped* exponentially in x, and it is *out of phase* with the forcing temperature on the boundary by an amount that depends on x. The maximum phase shift comes when $x = \pi\sqrt{2}$, so that

$$u(x,t) = e^{-\pi}\sin(t-\pi) = -e^{-\pi}\sin t$$

(and, of course, for other odd multiples of $\pi\sqrt{2}$). We note that $e^{-\pi} \approx 0.04$. Finally, suppose that x is depth in the earth and $u(0,t)$ is the surface temperature, which is roughly periodic with a period of one year. When one puts the physical units and constants back in, including a typical value for the thermal conductivity of the earth, it turns out that the maximum phase shift comes at $x \approx 13$ feet, a plausible depth to which to dig to make a food storage room if you live in a place or century without refrigeration.

Method 2: Laplace transform

What if $g(t)$ is not periodic? We might try to take a Fourier transform in time, but what if $g(t)$ does not fall off fast enough at ∞ to make the integral converge? One possibility is to take a Laplace transform, as one does with ODE initial-value problems.

Recall that if $F(s)$ is the Laplace transform of $f(t)$, then the transform of $f'(t)$ is $sF(s) - f(0)$. We can simplify our calculations by assuming $u(x,0) = 0$; as before, any other initial value can be handled by a solution of the homogenized problem. (Since this is not the same initial condition we used in the first method, the two solutions will not match exactly.)

Take Laplace transforms of the PDE and the boundary condition:

$$sU(x,s) = \frac{\partial^2 U}{\partial x^2}\,, \qquad U(0,s) = G(s).$$

The general solution is

$$U(x,s) = C_+ e^{x\sqrt{s}} + C_- e^{-x\sqrt{s}},$$

but we can discard the first term because it grows at infinity. Thus

$$U(x,s) = G(s)e^{-x\sqrt{s}}.$$

Now we can either struggle with the contour integral determining u from U (a matter for a course in complex variables) or find the answer in a good table of Laplace transforms, such as the one in the famous National Bureau of Standards *Handbook of Mathematical Functions* ... , edited by M. Abramowitz and I. Stegun. It tells us that the inverse transform of $e^{-x\sqrt{s}}$ is

$$\frac{xt^{-3/2}}{2\sqrt{\pi}}\, e^{-x^2/4t}.$$

Therefore, by the convolution theorem for Laplace transforms,

$$u(x,t) = \int_0^t \frac{xv^{-3/2}}{2\sqrt{\pi}}\, e^{-x^2/4v} g(t-v)\, dv.$$

With some effort, you can verify that this solution satisfies all the conditions. Verifying the PDE is a matter of differentiating under the integral sign and simplifying. The trickier part is checking that $u(0,t) = g(t)$, since at first glance it appears that $u(0,t) = 0$! What we have here is yet another representation of the delta function. Introduce a new variable, a, by

$$v = \frac{x^2}{4a}\,.$$

A short calculation shows that

$$u(x,t) = \int_{x^2/4t}^{\infty} \frac{a^{-1/2}e^{-a}}{\sqrt{\pi}}\, g\left(t - \frac{x^2}{4a}\right) da.$$

Assume that g is differentiable; then

$$g\left(t - \frac{x^2}{4a}\right) = g(t) - \frac{x^2}{4a}\, g'(t) + O(x^4);$$

better still, by the mean value theorem,

$$g\left(t - \frac{x^2}{4a}\right) = g(t) - \frac{x^2}{4a}\, g'(\theta)$$

exactly, for some θ. So, let's also make the reasonable assumption that g' is bounded. We now have

$$u(x,t) = g(t) \int_{x^2/4t}^{\infty} \frac{a^{-1/2} e^{-a}}{\sqrt{\pi}}\, da + \text{error},$$

$$|\text{error}| \le \frac{x^2}{4}\, \max |g'(\theta)| \int_{x^2/4t}^{\infty} \frac{a^{-3/2} e^{-a}}{\sqrt{\pi}}\, da.$$

It can be shown that

$$\int_0^{\infty} a^{-1/2} e^{-a}\, da \equiv \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

(see "gamma function" in the index of the Haberman or (better) Schaum's book). Thus the first term in the formula for $u(x,t)$ gives, in the limit $x \to 0$, just $g(t)$, which is what we want. The integral in the second term diverges as $x \to 0$, but slowly enough that the factor x^2 still drives that term to 0.