Math. 412 13 December 2004

# Final Examination – Solutions

## Some possibly useful information

Parseval equation for the Fourier series on $(-\pi, \pi)$:

$$\text{If} \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{then} \quad \int_{-\pi}^{\pi} |f(x)|^2 \, dx = 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2.$$

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \,.$$

Laplacian operator in spherical coordinates ("physicists' notation"):

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \phi^2} \,.$$

Spherical harmonics satisfy

$$\left[ \frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right] Y_l^m(\theta, \phi) = -l(l+1) Y_l^m(\theta, \phi).$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z} \frac{\partial Z}{\partial z} + \left( 1 - \frac{n^2}{z^2} \right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z).$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z} \frac{\partial Z}{\partial z} + \left( 1 - \frac{n(n+1)}{z^2} \right) Z = 0 \quad \text{has solutions } j_n(z) \text{ and } y_n(z).$$

Legendre's equation:

$$\frac{1}{\sin\theta} \frac{d}{d\theta} \left( \sin\theta \frac{d\Theta}{d\theta} \right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta).$$

Famous Green function integrals:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, e^{-k^2 t} \, dk = \frac{1}{\sqrt{4\pi t}} \, e^{-x^2/4t}, \qquad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, e^{-|k| y} \, dk = \frac{1}{\pi} \, \frac{y}{x^2 + y^2} \,.$$

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1. *(40 pts.)* Classify each equation as

   (i) linear homogeneous, linear nonhomogeneous, or nonlinear,

**and**

   (ii) elliptic, hyperbolic, or parabolic.

   (a) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-(x^2+y^2)} u^3 .$$

Nonlinear (because of the $u^3$); elliptic.

   (b) $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{e^{-t}}{x^2+1} .$$

Linear nonhomogeneous; parabolic. (It is a heat equation with a source.)

   (c) $$\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0.$$

Linear homogeneous; parabolic. Why is it parabolic? If we write the second-order terms as $a\partial_x{}^2 + b\partial_x\partial_y + c\partial_y{}^2$, then the discriminant $b^2 - 4ac = (-2)^2 - 4\cdot 1 = 0$. Another way to say it is that these leading terms are built out of a quadratic form with matrix $\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, which is obviously singular, so one of its eigenvalues is 0. Therefore, after diagonalization the second derivative with respect to one of the variables will not appear. (Indeed, if you define $t = x + y$ and $s = x - y$, a calculation shows that the equation transforms to

$$4\,\frac{\partial^2 u}{\partial s^2} + 2\,\frac{\partial u}{\partial t} = 0,$$

so it is a heat equation in disguise.)

   (d) $$\frac{1}{c^2}\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} .$$

Linear homogeneous; hyperbolic (wave equation).

2. *(40 pts.)* Solve Laplace's equation in a ball,

$$\nabla^2 u = 0 \quad \text{for} \quad 0 \le r < R, \qquad \frac{\partial u}{\partial r}(R,\theta,\phi) = f(\theta,\phi).$$

   (As usual, you may jump right to the answer if you know it.)

I will write the solution in terms of spherical harmonics. See Haberman, Sec. 7.10.6, for solution of a very similar problem (with different boundary condition) in terms of trig functions, Legendre functions, and $\theta \leftrightarrow \phi$.

After greater or lesser ado, you arrive at the general solution of Laplace's equation inside a ball,

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} C_{lm} r^l Y_l^m(\theta,\phi).$$

Thus

$$\frac{\partial u}{\partial r}(R,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} C_{lm}\, lR^{l-1} Y_l^m(\theta,\phi).$$

By definition, the spherical harmonics are orthonormal (on the *unit* sphere, so there's no extra factor $R^2$), so

$$C_{lm} = \frac{1}{lR^{l-1}} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_l^m(\theta,\phi)^* f(\theta,\phi).$$

(Units check: $C_{lm} r^l$ = pure number $\times\, R(r/R)^l\,[f]$, so $[u] = [\text{length}]\,[\partial u/\partial r]$, as it should be.)

If $l=0$ this formula has a problem; but that is just our old friend the solvability condition for the Neumann problem. For a solution to exist, one must have

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, Y_0^0(\theta,\phi)^* f(\theta,\phi) = \frac{1}{\sqrt{4\pi}} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\, f(\theta,\phi) = 0.$$

The solution is then nonunique, since any constant ($Y_0^0$ term) could be added without spoiling the boundary condition.

3. *(40 pts.)* By the method of your choice, solve the Laplace equation in a quadrant,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < \infty, \quad 0 < y < \infty,$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad u(x,0) = f(x).$$

(Require the solution to be bounded.)

*Method 1:* The Green function for Laplace's equation in the upper half plane is

$$G_0(x,z,y) = \frac{1}{\pi}\frac{y}{(x-z)^2+y^2}$$

(see "Famous Green function integrals" on first page of test). By the method of images, the Green function for the quadrant with Neumann boundary condition is

$$G(x,z,y) = G_0(x,z,y) + G_0(x,-z,y),$$

and so the solution of our problem is

$$u(x,y) = \int_0^{\infty} G(x,z,y) f(z)\, dz.$$

*Variant of Method 1:* The solution is the restriction to the quadrant of the solution in the whole upper half plane with boundary data equal to the even extension of $f$. Use $G_0$ to solve that problem.

*Method 2:* Separate variables as $u = X(x)Y(y)$.

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda.$$

We must have $X'(0) = 0$, which (together with boundedness) dictates $\lambda = k^2 > 0$, $X(x) = \cos(kx)$, $Y(y) = e^{-ky}$. Superposing,

$$u(x,y) = \int_0^\infty A(k)\cos(kx)e^{-ky}\,dk.$$

Then

$$f(x) = \int_0^\infty A(k)\cos(kx)\,dk, \quad A(k) = \frac{2}{\pi}\int_0^\infty f(x)\cos(kx)\,dx.$$

(Substituting $A$ into $u$ and evaluating the Famous Green function integral, you can reduce this answer to the result of Method 1, but that was not required.)

*Variant of Method 2:* From the homogeneous boundary condition, recognize immediately that a Fourier cosine transform with respect to $x$ is called for. Transform both the differential equation and the nonhomogeneous boundary condition, solve for the cosine transform of $u$, and invert the transform.

4. *(40 pts.)* *(You don't have to finish (a) to do the later parts!)*

(a) I claim that the ("full") Fourier series of $f(x) = x^2$ on the interval $(-\pi, \pi)$ is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{with} \quad c_0 = \frac{\pi^2}{3}, \quad c_n = \frac{2(-1)^n}{n^2} \text{ for } n \neq 0.$$

Write the coefficient formula, calculate $c_0$, and describe in words how you would calculate $c_n$ if you had time.

The coefficient formula is

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}x^2\,dx.$$

Thus

$$c_0 = \frac{1}{2\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{2\pi^3}{2\pi\cdot 3}.$$

The other coefficients require more work than we have time for:

*Method 1:* The indefinite integral must have the form

$$\int e^{-inx}x^2\,dx = e^{-inx}(Ax^2 + Bx + C).$$

Differentiate, get equations to solve for $A$, $B$, and $C$. (It turns out that only $B$ contributes to the definite integral.)

*Method 2:* Integrate by parts twice.

(b) Discuss the convergence of the series and sketch the limit function over the interval $-2\pi < x < 4\pi$.

The series converges to the periodic extension of $f$ to the whole real line. The extension is continuous as well as piecewise smooth, so the convergence is uniform. Except for the range of the variable, the graph should look like the scalloped curve on p. 20 of the notes (in the paragraph headed "*Caution*").

(c) Use (a) to evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

To get $n^{-4}$ from this series, we must be using Parseval's equation (see top of first page of test).

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{10\pi}\, x^5\Big|_{-\pi}^{\pi} = \frac{\pi^4}{5}.$$

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

Equate these and solve for the desired sum:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}.$$

(d) Evaluate (a) at a particular value of $x$ to show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$.

Try $x = 0$:

$$0 = \sum_{n=-\infty}^{\infty} c_n = c_0 + 2\sum_{n=1}^{\infty} c_n = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Solve for the desired sum:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

5. *(40 pts.)* Let $V(r)$ be a continuous function (independent of $\theta$) and consider the eigenvalue problem in polar coordinates

$$-\nabla^2 u + V(r)u = \lambda u \quad \text{for} \quad 1 < r < 2, \quad 0 < \theta \le 2\pi,$$

periodic boundary conditions in $\theta$,

$$u(1,\theta) = 0 = u(2,\theta).$$

(a) Solve the equation by separation of variables, discovering a new class of special functions of $r$ (which you may name after yourself). Say what you can on the basis of general principles about those functions and about the eigenvalues $\lambda$.

Try a separated solution $u = R(r)\Theta(\theta)$. We have

$$-R''\Theta - \frac{1}{r}R'\Theta - \frac{1}{r^2}R\Theta'' + VR\Theta = \lambda R\Theta.$$

Rearranging,

$$-\frac{\Theta''}{\Theta} = n^2 = r^2\frac{R''}{R} + r\frac{R'}{R} + (\lambda - V)r^2,$$

where we know that $n$ is an integer because of the periodic boundary conditions. (We can write the angular part of the solutions either as $e^{in\theta}$ with $-\infty < n < \infty$ or as $\sin(n\theta)$ $(1 \le n < \infty)$ and $\cos(n\theta)$ $(0 \le n < \infty)$. Since the emphasis in this problem is on the radial functions, I'll say no more about the angular ones.)

Rearranging once more, we get

$$R'' + \frac{1}{r}R' + \left(\lambda - V(r) - \frac{n^2}{r^2}\right)R = 0,$$

subject to the boundary conditions $R(1) = 0 = R(2)$. This is a classic regular Sturm–Liouville problem. There will be infinitely many eigenvalues $\lambda_n$ $(\lambda_1 < \lambda_2 < \cdots)$, all real, and the corresponding eigenfunctions $R_n(r)$ will be orthogonal with respect to the integration element $r\,dr$. $R_n$ has $n-1$ nodes inside the interval. The eigenfunctions form a complete set (a basis for the Hilbert space $L^2(1,2)$ with respect to "convergence in the mean").

(b) State a condition on $V(r)$ that will guarantee that all the eigenvalues are positive.

$V(r) \ge 0$ for all $r \in (1,2)$ will do. $V(r) \ge -n^2/r^2$ is even better (less restrictive). Proof ("Rayleigh quotient"): Assuming that you have a solution, multiply the equation by $R$ and integrate:

$$\lambda\int_1^2 R(r)^2\,r\,dr = -\int_1^2 r\,dr\,\left[R'' + \frac{1}{r}R' - \left(V(r) + \frac{n^2}{r^2}\right)R\right]R(r).$$

Integrate by parts in the first term on the right:

$$\lambda\int_1^2 R(r)^2\,r\,dr = +\int_1^2 r\,dr\,\left[R'(r)^2 + \left(V(r) + \frac{n^2}{r^2}\right)R(r)^2\right].$$

(The endpoint terms vanished because of the boundary conditions, and the $R'R$ terms cancelled.) It is clear that all the terms are positive with the possible exception of the one containing $V(r)$ (and the $n^2$ term if $n = 0$). If $V$ is nonnegative (or even if it's occasionally negative but so small that it doesn't outweigh the other terms), then $\lambda$ will be positive.

(c) Be more specific about what happens in the case that $V(r) = 0$ for all $r$.

To simplify notation I assume $n \ge 0$. $R(r)$ is some linear combination of the Bessel functions $J_n(\omega r)$ and $Y_n(\omega r)$, where $\omega = \sqrt{\lambda}$. The allowed values of $\omega$ are those that allow some such combination to vanish at both $r = 1$ and $r = 2$. They are the solutions of

$$\begin{vmatrix} J_n(\omega) & Y_n(\omega) \\ J_n(2\omega) & Y_n(2\omega) \end{vmatrix} = 0.$$

Then $R(r) = AJ_n(\omega r) + BY_n(\omega r)$ where

$$\frac{A}{B} = -\frac{Y_n(\omega)}{J_n(\omega)} = -\frac{Y_{2n}(\omega)}{J_{2n}(\omega)}.$$

(It might happen that $J_n(\omega) = 0 = J_{2n}(\omega)$, in which case $B = 0$.)