Math. 412 (Fulling) 22 October 2004

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

Possibly useful integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

1. *(35 pts.)* Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty, \quad 0 < t < \infty),$$

$$u(0,t) = T \quad \text{(a positive constant)}, \qquad u(x,0) = f(x).$$

(Require the solution to be bounded as $x \to +\infty$.) Extra credit can be obtained by

(A) simplifying the solution to a form involving a Green function;

(B) making a wise comment upon the most natural condition (mathematically) to impose on f as $x \to +\infty$.

There are two nonhomogeneous boundary conditions, so we expect to split the solution into a sum of two solutions of the PDE. The time-dependent boundary function is a constant, which suggests looking for a steady-state solution as the first step. Such a solution must satisfy

$$\frac{d^2 v}{dx^2} = 0, \qquad v(0) = T.$$

Therefore, $v(x) = Ax + B$; the boundary condition forces $B = T$, and boundedness requires $A = 0$.

Now the other part of the solution, $w = u - v$, satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}\,, \qquad w(0,t) = 0, \qquad w(x,0) = f(x) - T \equiv g(x).$$

The data on the infinite interval $(0,\infty)$ dictates a Fourier transform of some kind, and the boundary condition at $x = 0$ tells us that this should be a sine transform. In other words, the separated solutions will be of the form $\sin(kx)T(t)$, and then the PDE says $\sin(kx)T'(t) = -k^2 \sin(kx)T(t)$, hence $T(t) = e^{-k^2 t}$. (Of course, this T is not the same one as in the temperature boundary condition.) So we write the solution as

$$w(x,t) = \int_0^\infty B(k)\sin(kx)e^{-k^2 t}\, dk.$$

Then

$$g(x) = \int_0^\infty B(k)\sin(kx)\, dk,$$

so

$$B(k) = \frac{2}{\pi} \int_0^\infty g(y) \sin(ky)\, dy.$$

In summary,

$$u(x,t) = \int_0^\infty B(k) \sin(kx) e^{-k^2 t}\, dk + T$$

where

$$B(k) = \frac{2}{\pi} \int_0^\infty \bigl(f(y) - T\bigr) \sin(ky)\, dy.$$

(A): The Green function will be relevant only to the w part of the solution:

$$w(x,t) = \int_0^\infty G(x,y,t) g(y)\, dy.$$

Substituting the B formula into the previous w formula, we get

$$G(x,y,t) = \frac{2}{\pi} \int_0^\infty \sin(kx) \sin(ky) e^{-k^2 t}\, dk.$$

No need to stop there:

$$\begin{aligned} G(x,y,t) &= \frac{2}{\pi} \frac{1}{(2i)^2} \int_0^\infty \left[e^{ikx+iky} - e^{ikx-iky} - e^{-ikx+iky} + e^{-ikx-iky} \right] e^{-k^2 t}\, dk \\ &= -\frac{1}{2\pi} \int_{-\infty}^\infty \left[e^{ik(x+y)-k^2 t} - e^{ik(x-y)-k^2 t} \right] dk \\ &= \frac{1}{\sqrt{4\pi t}} \left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t} \right]. \end{aligned}$$

This Green function for the Dirichlet heat problem on the half line is also obtained instantly by the method of images from the standard Green function for the heat problem on the whole line.

(B): To have a nice Fourier sine transform B, we would expect g to be in a nice function class, such as L^2 (square-integrable). Thus if f approaches a limit at all, it should approach T (not 0). This conclusion may seem surprising, since physically one might be more comfortable with $f \to 0$. It is an indication that this idealized problem is slightly singular; how long would it take for an infinitely long bar to come to thermal equilibrium with a heat source at one end? (Closer examination shows that if $f(x) \to 0$ as $x \to \infty$, then $B(k)$ has a singularity proportional to $1/k$ at the origin (like the Fourier transform of the Heaviside step function). This is something we could live with in the formula for the inverse sine transform. Also, the Green-function formula for w still converges, even if $g(y) \to T \neq 0$.)

2. *(30 pts.)*

(a) Construct a Green function to solve the ODE problem

$$\frac{d^2y}{dx^2} + 4y = f(x), \qquad y(0) = 0, \quad \frac{dy}{dx}(L) = 0.$$

(That is, find G so that $y(x) = \int_0^L G(x,z) f(z)\, dz$.)

The Green function should satisfy

$$\frac{\partial^2 G}{\partial x^2} + 4G = \delta(x-z), \qquad G(0,z) = 0, \quad \frac{\partial G}{\partial x}(L,z) = 0.$$

Therefore,

$$G(x,z) = \begin{cases} A\sin(2x) & \text{for } x < z, \\ B\cos[2(x-L)] & \text{for } x > z. \end{cases}$$

(We have already built the boundary conditions into the solution, so we have only two remaining constants to find, instead of four.) Now require that the solution be continuous,

$$G(z,z) = A\sin(2z) = B\cos[2(z-L)],$$

and that it satisfy the proper jump condition,

$$1 = \left.\frac{\partial G}{\partial x}\right|_{x=z+\epsilon} - \left.\frac{\partial G}{\partial x}\right|_{x=z-\epsilon} = -2B\sin[2(z-L)] - 2A\cos(2z).$$

Rewrite the system neatly:

$$\begin{aligned} A\sin(2z) - B\cos[2(z-L)] &= 0, \\ A\cos(2z) + B\sin[2(z-L)] &= -\tfrac{1}{2}\,. \end{aligned}$$

Such a system is most easily solved by Cramer's rule. The basic determinant is

$$\Delta = \begin{vmatrix} \sin(2z) & -\cos[2(z-L)] \\ \cos(2z) & \sin[2(z-L)] \end{vmatrix} = \sin(2z)\sin[2(z-L)] + \cos(2z)\cos[2(z-L)] = \cos(2L).$$

Then

$$A = \frac{1}{\Delta}\begin{vmatrix} 0 & -\cos[2(z-L)] \\ -\frac{1}{2} & \sin[2(z-L)] \end{vmatrix} = -\frac{\cos[2(z-L)]}{2\cos(2L)}\,,$$

$$B = \frac{1}{\Delta}\begin{vmatrix} \sin(2z) & 0 \\ \cos(2z) & -\frac{1}{2} \end{vmatrix} = -\frac{\sin(2z)}{2\cos(2L)}\,.$$

So, finally,

$$G(x,z) = \begin{cases} -\frac{\sin(2x)\cos[2(z-L)]}{2\cos(2L)} & \text{for } x < z, \\ -\frac{\cos[2(x-L)]\sin(2z)}{2\cos(2L)} & \text{for } x > z. \end{cases}$$

(b) For what values of L is this problem impossible? Explain why those values are special.

The algebraic system has no solution when $\Delta = 0$, which occurs if $2L = \left(n + \frac{1}{2}\right)\pi$ (in other words, L is an odd multiple of $\pi/4$). These are the lengths for which the *homogeneous* differential equation has nontrivial solutions, so that the solution of the nonhomogeneous equation is not unique (and may not even exist for some f s). That is, for such L the two endpoint solutions, $\sin(2x)$ and $\cos[2(x - L)]$, are *equal* up to a constant factor, so that $\sin(2x)$ is an *eigenfunction* of the homogeneous problem ($X'' = \lambda X$, $X(0) = 0$, $X'(L) = 0$) with eigenvalue $\lambda = -4$.

3. *(35 pts.)* Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (0 < x < K, \quad 0 < y < L),$$

$$u(0, y, t) = 0 = u(K, y, t), \qquad \frac{\partial u}{\partial y}(x, 0, t) = 0 = \frac{\partial u}{\partial y}(x, L, t),$$

$$u(x, y, 0) = f(x, y).$$

Feel free to skip routine steps if you are sure you know the answer.

This is very similar to a problem worked out in the class notes.