Math. 412 (Fulling) 19 November 2004

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(55 pts.)*

   (a) Solve Laplace's equation, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, in a square, $0 < x < \pi$ and $0 < y < \pi$, with the boundary conditions

$$u(0,y) = 0\,, \quad u(\pi,y) = f(y)\,, \qquad u(x,0) = 0\,, \quad \frac{\partial u}{\partial y}(x,\pi) + 2u(x,\pi) = 0\,.$$

   In the process you will discover a sequence of eigenfunctions and eigenvalues, which you should name $\{\phi_n(y)\}$ and $\{{\omega_n}^2\}$. Describe the ω_n qualitatively but don't expect to find their exact numerical values. (Also, don't bother to evaluate the normalization integral.)

Look for separated solutions $X(x)Y(y)$ (postponing the nonhomogeneous BC).

$$\frac{X''}{X} = -\frac{Y''}{Y} = \omega^2.$$

Assuming for the moment that $\omega^2 > 0$, the two problems are

$$X'' = \omega^2 X, \quad X(0) = 0 \qquad \text{— hence } X(x) = \sinh(\omega x)\,,$$

$$Y'' = -\omega^2 Y, \quad Y(0) = 0, \quad Y'(\pi) + 2Y(\pi) = 0\,.$$

From this we get

$$Y(y) = \sin(\omega y), \qquad -\frac{1}{2}\,\omega = \tan(\pi\omega)\,.$$

Graphing the two sides of the eigenvalue equation (for positive ω), we see that there are intersections on each branch of the tangent function except the first. That is, there are eigenvalues ${\omega_n}^2$, $n = 1, 2, \ldots$, with ω_n slightly greater than $(n + \frac{1}{2})\pi$. Let's write $Y_n(y) = \sin(\omega_n y)$ and $\phi_n(y)$ for the normalized eigenfunctions, $Y_n(y)/\|Y_n\|$. (Here

$$\|Y_n\|^2 = \int_0^\pi Y_n(y)^2\, dy\,,$$

and you were not expected to evaluate this integral.)

The general theorem about Sturm–Liouville problems (with $+2$ in the Robin boundary condition at the top end of the interval) guarantees that negative and zero eigenvalues don't occur. This can also be shown directly.

We can now write the solution of the original problem as

$$u(x,y) = \sum_{n=1}^{\infty} c_n \phi_n(y) \sinh(\omega_n x)$$

and impose the boundary data

$$f(y) = \sum_{n=1}^{\infty} c_n \phi_n(y) \sinh(\omega_n \pi) \, .$$

Since the basis functions are normalized, we can calculate immediately

$$c_n = \frac{\int_0^\pi \phi_n(y) f(y)\, dy}{\sinh(\omega_n \pi)} \, .$$

Alternatively, in terms of the Y_n you have

$$u(x,y) = \sum_{n=1}^{\infty} C_n Y_n(y) \sinh(\omega_n x)$$

and

$$C_n = \frac{\int_0^\pi Y_n(y) f(y)\, dy}{\|Y_n\|^2 \sinh(\omega_n \pi)} \, .$$

(b) Express the solution in terms of a Green function, so that

$$u(x,y) = \int_0^\pi G(x,y,z) f(z)\, dz,$$

with a formula for G in terms of the eigenfunctions ϕ_n.

Write the c_n integral with the variable of integration called z instead of y, then insert into the formula for u. You arrive at

$$G(x,y,z) = \sum_{n=1}^{\infty} \frac{\sinh(\omega_n x)}{\sinh(\omega_n \pi)} \phi_n(y) \phi_n(z) \, .$$

If you use Y instead of ϕ, you need to divide by $\|Y_n\|^2$.

(c) What are the orthogonality and completeness relations satisfied by the eigenfunctions? (You can answer (c) on abstract grounds even if you have trouble with (a).)

$$\int_0^\pi \phi_n(y) \phi_m(y)\, dy = \delta_{nm} \, , \qquad \sum_{n=1}^{\infty} \phi_n(y) \phi_n(z) = \delta(y-z) \, .$$

(Since the eigenfunctions are real, no complex conjugations are necessary, but it doesn't hurt to put them in.) In terms of Y,

$$\int_0^\pi Y_n(y) Y_m(y)\, dy = \delta_{nm} \|Y_n\|^2 \, , \qquad \sum_{n=1}^{\infty} \frac{Y_n(y) Y_n(z)}{\|Y_n\|^2} = \delta(y-z) \, .$$

A check: Set $x = \pi$ in the Green function formula:

$$\delta(y-z) = G(\pi, y, z) = \sum_{n=1}^{\infty} \phi_n(y) \phi_n(z) \, .$$

2. *(45 pts.)* Solve the wave equation for a circular (but warped) drum of radius 5 (i.e., $0 \le r < 5$ and $0 \le \theta < 2\pi$) with boundary data

$$u(5,\theta,t) = \sin\theta$$

and initial data

$$u(r,\theta,0) = f(r,\theta)\,, \qquad \frac{\partial u}{\partial t}(r,\theta,0) = 0\,.$$

The equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

You may skip routine steps if you're sure you know the form of the answer. *Hint:* First find a steady-state solution.

Since there are two nonhomogeneous boundary conditions, involving boundaries of different types, we need to split the problem. The lack of t dependence in the first BC suggests solving first the steady-state problem,

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0\,, \quad v(5,\theta,t) = \sin\theta\,.$$

The solutions of Laplace's equation inside a disk are known to be superpositions of the modes $r^n\sin(n\theta)$ and $r^n\cos(n\theta)$ (or $r^n e^{\pm in\theta}$). In the present case we see that we need only one term, not a whole Fourier series (two terms if the $e^{\pm in\theta}$ modes are used):

$$v(r,\theta) = \frac{r}{5}\sin\theta\,.$$

Now let $w = u - v$. It must satisfy

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\,,$$

$$w(5,\theta,t) = 0, \quad w(r,\theta,0) = h(r,\theta) \equiv f(r,\theta) - v(r,\theta)\,, \quad \frac{\partial w}{\partial t}(r,\theta,0) = 0\,.$$

This time the normal modes will be $J_n(\omega r)e^{\pm in\theta}$. The boundary condition becomes $J_n(5\omega) = 0$, which requires that $\omega = \frac{1}{5}z_{nj}$, z_{nj} being the jth zero of the Bessel function J_n . The time dependence is $\cos(\omega t)$ since $\frac{\partial u}{\partial t}$ must vanish at $t = 0$. All together, then,

$$w(r,\theta,t) = \sum_{\nu=-\infty}^{\infty}\sum_{j=1}^{\infty} c_{\nu j} e^{i\nu\theta} J_{|\nu|}(z_{|\nu|j}/5)\cos(z_{|\nu|j}t/5)\,.$$

Finally,

$$h(r,\theta) = \sum_{\nu=-\infty}^{\infty}\sum_{j=1}^{\infty} c_{\nu j} e^{i\nu\theta} J_{|\nu|}(z_{|\nu|j}/5)\,,$$

whence

$$c_{\nu j} = \frac{\int_0^{2\pi}\int_0^5 e^{-i\nu\theta} J_{|\nu|}(z_{|\nu|j}/5) h(r,\theta)\, r\,dr\,d\theta}{2\pi\int_0^5 J_{|\nu|}(z_{|\nu|j}/5)^2\, r\,dr}\,.$$