Math. 412 9 December 2005

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Some possibly useful information

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\,\frac{\partial u}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 u}{\partial \theta^2}\,.$$

Laplacian operator in spherical coordinates ("physicists' notation"):

$$\nabla^2 u = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2 u}{\partial\phi^2}\,.$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right] Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi).$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\,\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z).$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z}\,\frac{\partial Z}{\partial z} + \left(1 - \frac{l(l+1)}{z^2}\right) Z = 0 \quad \text{has solutions } j_l(z) \text{ and } y_l(z).$$

Legendre's equation:

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta).$$

Famous Green function integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,e^{-k^2 t}\,dk = \frac{1}{\sqrt{4\pi t}}\,e^{-x^2/4t}, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,e^{-|k|y}\,dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

Hyperbolic function identities:

$\sinh(A \pm B) = \sinh A \cosh B \pm \cosh A \sinh B, \qquad \cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B.$

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1. *(30 pts.)* Classify each equation as

   (i) elliptic, hyperbolic, or parabolic,

and

   (ii) linear homogeneous, linear nonhomogeneous, or nonlinear.

(a) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

Parabolic; linear homogeneous.

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 2e^{2u}$

Elliptic; nonlinear.

(c) $\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} + 2c\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$

Hyperbolic; linear homogeneous.

2. *(35 pts.)* Solve Laplace's equation in a disk,

$$\nabla^2 u = 0 \quad \text{for} \quad 0 \le r < R, \qquad u(R,\theta) = f(\theta).$$

Since this problem is done in detail in my notes and in every textbook on the subject, I will be brief here. The general solution inside the disk is

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}.$$

Then the boundary condition is

$$f(\theta) = \sum_{n=-\infty}^{\infty} c_n R^{|n|} e^{in\theta},$$

whence

$$c_n = R^{-|n|} \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta.$$

3. *(35 pts.)*

   (a) Construct a Green function to solve the nonhomogeneous ODE problem

$$\frac{\partial^2 u}{\partial x^2} - \kappa^2 u = f(x) \quad (0 < x < 1), \qquad u'(0) = 0, \quad u'(1) = 0,$$

   for $\kappa > 0$.

The Green function should satisfy

$$\frac{\partial^2 G}{\partial x^2} - \kappa^2 G = \delta(x-y), \qquad \frac{\partial G}{\partial x}(0,y) = 0 = \frac{\partial G}{\partial x}(1,y).$$

We must have

$$G(x,y)=\begin{cases} a(y)e^{\kappa x}+b(y)e^{-\kappa x} & \text{for } x<y,\\ c(y)e^{\kappa x}+d(y)e^{-\kappa x} & \text{for } x>y.\end{cases}$$

The two boundary conditions imply

$$a(y)-b(y)=0, \quad c(y)e^{\kappa}-d(y)e^{-\kappa}=0,$$

whence (as you might have had foresight to see from the start)

$$G(x,y)=\begin{cases} A(y)\cosh(\kappa x) & \text{for } x<y,\\ B(y)\cosh(\kappa(x-1)) & \text{for } x>y.\end{cases}$$

[Details: $d(y)=c(y)e^{2\kappa} \Rightarrow c(y)e^{\kappa x}+d(y)e^{-\kappa x}=c(y)[e^{\kappa x}+e^{2\kappa}e^{-\kappa x}]=c(y)e^{\kappa}\cosh(\kappa x-\kappa)$, so let $B(y)=c(y)e^{\kappa}$.] Now impose the continuity and jump conditions at $x=y$:

$$A(y)\cosh(\kappa y)=B(y)\cosh(\kappa(y-1)),$$

$$\frac{\partial G}{\partial x}(y+\epsilon)-\frac{\partial G}{\partial x}(y-\epsilon)=1 \Rightarrow B(y)\sinh(\kappa(y-1))-A(y)\sinh(\kappa y)=\frac{1}{\kappa}.$$

Let's rewrite them neatly:

$$\begin{aligned} A(y)\cosh(\kappa y)-B(y)\cosh(\kappa(y-1))&=0,\\ -A(y)\sinh(\kappa y)+B(y)\sinh(\kappa(y-1))&=\frac{1}{\kappa}.\end{aligned}$$

The determinant of this system is

$$\Delta=\begin{vmatrix}\cosh(\kappa y) & -\cosh(\kappa(y-1))\\ -\sinh(\kappa y) & \sinh(\kappa(y-1))\end{vmatrix}=\cosh(\kappa y)\sinh(\kappa(y-1))-\sinh(\kappa y)\cosh(\kappa(y-1))=-\sinh\kappa.$$

Thus

$$\begin{aligned} A(y)&=\frac{-1}{\sinh\kappa}\begin{vmatrix}\cosh(\kappa y) & 0\\ \kappa^{-1} & \sinh(\kappa(y-1))\end{vmatrix}=\frac{-\cosh(\kappa(y-1))}{\kappa\sinh\kappa},\\ B(y)&=\frac{-1}{\sinh\kappa}\begin{vmatrix}0 & -\cosh(\kappa(y1-))\\ -\sinh(\kappa(y-1)) & \kappa^{-1}\end{vmatrix}=\frac{-\cosh(\kappa y)}{\kappa\sinh\kappa}.\end{aligned}$$

Therefore,

$$G(x,y)=\begin{cases} -\frac{\cosh(\kappa x)\cosh(\kappa(y-1))}{\kappa\sinh\kappa} & \text{for } x<y,\\ -\frac{\cosh(\kappa y)\cosh(\kappa(x-1))}{\kappa\sinh\kappa} & \text{for } x>y.\end{cases}$$

(b) Explain why there is no Green function in the case $\kappa=0$. *Hint:* Qu. 5, Answer (B) arises in an analogous situation.

When $\kappa=0$ any constant function is a solution of the homogeneous problem,

$$\frac{\partial^2 u}{\partial x^2}-\kappa^2 u=0, \qquad u'(0)=0, \quad u'(1)=0.$$

Therefore, the solution of the nonhomogeneous problem is not unique. In fact, a solution will not even exist unless $\int_0^1 f(x)\,dx=0$. Therefore, no formula of the Green type can apply to that problem.

4. *(35 pts.)* Solve the heat equation in a ball,

$$\frac{\partial u}{\partial t} = \nabla^2 u \quad \text{for} \quad 0 \le r < 1, \qquad u(t,1,\theta,\phi) = 0, \quad u(0,r,\theta,\phi) = 1 - r).$$

(As usual, you may skip well-known steps if you know where you're heading and can explain what you're doing.)

Separate variables as $u = U_\omega(r,\theta,\phi)e^{-\omega^2 t}$, arriving at $-\omega^2 U_\omega = \nabla^2 U_\omega$, the Laplacian in spherical coordinates being given on the first page of the test.

The quickest way to proceed is to notice that since the data function in this problem is independent of the angles, the relevant eigenfunctions will be, too; therefore, we can discard all the angular derivatives and get

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -\omega^2 U.$$

Let $z = \omega r$; then

$$\frac{d^2U}{dz^2} + \frac{2}{z}\frac{dU}{dz} + U = 0,$$

which is the spherical Bessel equation with $l = 0$. The solution that is regular at the origin is $j_0(z)$. We need the eigenfunction to vanish when $R = 1$, so the allowed values of ω are the zeros of $j_0(z)$. So the solution of the main problem has the form

$$u(t,r,\theta,\phi) = \sum_{j=1}^{\infty} C_j\, j_0(\omega_j r)e^{-\omega_j{}^2 t}.$$

A more systematic way of getting to this point is to write the general solution

$$u(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{j=1}^{\infty} C_{lmj} Y_l^m(\theta,\phi) j_l(\omega_{lj} r)e^{-\omega_j{}^2 t}$$

and later observe that all the spherical harmonics except Y_0^0 (a constant function) are orthogonal to the initial data.

Finally, we should find formulas for the C_j. Recall that the spherical Bessel functions solve a Sturm–Liouville problem with weight function r^2. So

$$C_j = \frac{\int_0^1 j_0(\omega_j r)(1-r)r^2\,dr}{\int_0^1 j_0(\omega_j r)^2 r^2\,dr}\,.$$

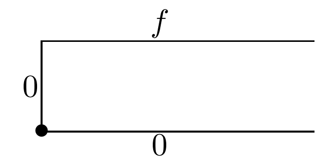
Although it may be possible to evaluate these integrals, we won't try.

5. *(30 pts.)* This is a group of multiple-choice questions, concerning the equation $\nabla^2 u = 0$ in various regions of the x-y plane with various boundary conditions. For each part of the problem, select from the following list the form that you expect the solution to have. *You should be able to answer these questions from a knowledge of general principles and a few moments' thought, without calculations. It might help to work* **from** *the answers* **to** *the questions.*

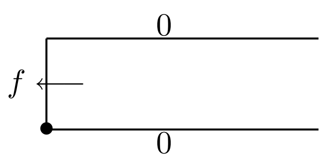
Please note that answers are labeled by *capital* letters, and questions by *lower case* letters. Please list your responses in alphabetical order of the *questions*: something like "a — B, b — G, [etc]". (One answer is not used. No answer is correct to more than one question.)

Here are the questions. The large dot in each figure indicates the origin of coordinates. Whenever a region has a side of finite length, that length is L. An open end indicates the region extends to infinity. An arrow across the boundary indicates a normal-derivative (a.k.a. Neumann or flux) boundary condition; otherwise the boundary data apply to u itself (Dirichlet condition). "f" represents arbitrary inhomogeneous data.

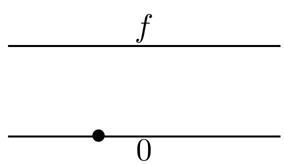
(a)



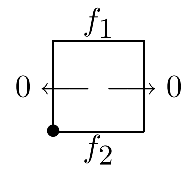
(b)



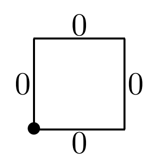
(c)



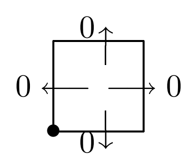
(d)



(e)



(f)



And here are the allowed answers:

(A) $u(x,y) = 0$

(B) $u(x,y) = C$ (a nonzero constant)

(C) $u(x,y) = \displaystyle\int_0^\infty d\omega\, B(\omega)\, \sin(\omega x)\, \sinh(\omega y)$

(D) $u(x,y) = \displaystyle\sum_{n=1}^\infty \sin\frac{n\pi y}{L}\left[A_n \sinh\frac{n\pi x}{L} + B_n \sinh\frac{n\pi(L-x)}{L}\right]$

(E) $u(x,y) = \displaystyle\sum_{n=1}^\infty C_n \sin\frac{n\pi y}{L}\, e^{-n\pi x/L}$

(F) $u(x,y) = \displaystyle\int_{-\infty}^\infty d\omega\, A(\omega)\, e^{i\omega x}\, \sinh(\omega y)$

(G) $u(x,y) = \displaystyle\sum_{n=0}^\infty \cos\frac{n\pi x}{L}\left[A_n \sinh\frac{n\pi y}{L} + B_n \sinh\frac{n\pi(L-y)}{L}\right]$

And here are the actual answers:

a — C
b — E
c — F
d — G
e — A
f — B
unused — D

6. *(35 pts.)* By the method of your choice, solve the wave equation on the half-line,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty < t < \infty, \quad 0 < x < \infty,$$

with conditions

$$\frac{\partial u}{\partial x}(t,0) = 0, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x).$$

(Require the solution to be bounded.)

Fourier's method: By separation of variables or an immediate Fourier cosine transform, the solution has the form

$$u(t,x) = \int_0^\infty dk\, A(k)\cos(kx)\cos(kt) + \int_0^\infty dk\, B(k)\cos(kx)\sin(kt).$$

From the initial data,

$$f(x) = \int_0^\infty dk\, A(k)\cos(kx), \qquad g(x) = \int_0^\infty dk\, B(k)\cos(kx)\, k.$$

So

$$A(k) = \frac{2}{\pi}\int_0^\infty dx\, f(x)\cos(kx), \qquad B(k) = \frac{2}{\pi k}\int_0^\infty dx\, g(x)\cos(kx).$$

D'Alembert's method: Define f and g for negative x to be the even extensions of the functions given for positive x. Define $G(x) = \int_0^x g(u)\,du$. Then

$$\begin{aligned} u(t,x) &= \tfrac{1}{2}[f(x+t) + f(x-t) + G(x+t) - G(x-t)] \\ &= \frac{1}{2}\left[f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(u)\,du\right]. \end{aligned}$$

Extra Credit Problems (30 points each)

7. Solve the wave problem (Qu. 6) by a distinctly different method.

[See above.]

8. Construct a Green function implementing the solution of the disk problem (Qu. 2). (The answer is known as "Poisson's formula".)

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta} R^{-|n|} \frac{1}{2\pi} \int_0^{\pi} e^{-in\phi} f(\phi)\, d\phi = \int_0^{2\pi} d\phi\, f(\phi) G(\theta,\phi)$$

where

$$\begin{aligned}
G(\theta,\phi) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{r}{R}\right)^{|n|} e^{in(\theta-\phi)} \\
&= \frac{1}{2\pi} \left[1 + \sum_{n=1}^{\infty} \left(\frac{r}{R}\right)^{n} (e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}) \right] \\
&= \frac{1}{2\pi} \left[1 + \frac{\frac{r}{R} e^{i(\theta-\phi)}}{1 - \frac{r}{R} e^{i(\theta-\phi)}} + \frac{\frac{r}{R} e^{-i(\theta-\phi)}}{1 - \frac{r}{R} e^{-i(\theta-\phi)}} \right] \\
&= \frac{1}{2\pi} \left[1 + \frac{\frac{r}{R} e^{i(\theta-\phi)} - \left(\frac{r}{R}\right)^2 + \frac{r}{R} e^{-i(\theta-\phi)} - \left(\frac{r}{R}\right)^2}{1 - \frac{2r}{R} \cos(\theta-\phi) + \left(\frac{r}{R}\right)^2} \right] \\
&= \frac{1}{2\pi} \left[1 + \frac{\frac{2r}{R} \cos(\theta-\phi) - 2\left(\frac{r}{R}\right)^2}{1 - \frac{2r}{R} \cos(\theta-\phi) + \left(\frac{r}{R}\right)^2} \right] \\
&= \frac{1}{2\pi} \frac{1 - \left(\frac{r}{R}\right)^2}{1 - \frac{2r}{R} \cos(\theta-\phi) + \left(\frac{r}{R}\right)^2} \\
&= \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2} \,.
\end{aligned}$$