Math. 412 (Fulling) 8 November 2005

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

Famous integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,e^{-k^2t}\,dk = \frac{1}{\sqrt{4\pi t}}\,e^{-x^2/4t}\,, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,e^{-|k|y}\,dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

1. *(20 pts.)* Use Parseval's equation to evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+4)^2}$.

Parseval's equation for Fourier transforms is

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \int_{-\infty}^{\infty}|\hat{f}(k)|^2\,dk.$$

Let $y=2$ in the second famous integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,e^{-2|k|}\,dk = \frac{1}{\pi}\,\frac{2}{x^2+4}\,.$$

Therefore, if

$$f(x) = \frac{1}{x^2+4} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)\,dk\,,$$

then

$$\hat{f}(k) = \frac{\pi}{2}\,\sqrt{2\pi}\,\frac{1}{2\pi}e^{-2|k|} = \frac{\sqrt{2\pi}}{4}\,e^{-2|k|}.$$

Thus

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{2\pi}{16}\int_{-\infty}^{\infty} e^{-4|k|}\,dk = \frac{4\pi}{16}\int_{0}^{\infty} e^{-4k}\,dk = \frac{\pi}{4}\,\left.\frac{e^{-4k}}{-4}\right|_0^{\infty} = \frac{\pi}{16}\,.$$

Check: (If you have had a complex analysis course, you know Cauchy's integral formula.)

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+4)^2} = 2\pi i\,\frac{d}{dx}\,\left.\frac{1}{(x+2i)^2}\right|_{x=2i} = \left.\frac{-4\pi i}{(x+2i)^3}\right|_{x=2i} = \frac{-4\pi i}{(4i)^3} = \frac{\pi}{16}\,.$$

2. *(45 pts.)* Solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < t < \infty, \quad 0 < x < 1),$$

$$\frac{\partial u}{\partial x}(t,0) = 0, \qquad \frac{\partial u}{\partial x}(t,1) = -2u(t,1),$$

$$u(0,x) = f(x).$$

Recommended strategy: Indicate graphically how to find the eigenvalues involved. Introduce a suitable notation for the eigenfunctions, and (required) **state the orthogonality and completeness relations** satisfied by those functions. Then finish the solution.

First look for separated solutions, $u = X(x)T(t)$. As usual we get (assuming for the moment that all the eigenvalues are positive)

$$\frac{T'}{T} = \frac{X''}{X} = -\omega^2, \quad \text{so} \quad T = e^{-\omega^2 t}.$$

Then

$$X'' = -\omega^2 X, \quad X'(0) = 0 \quad \text{imply} \quad X \propto \cos(\omega x),$$

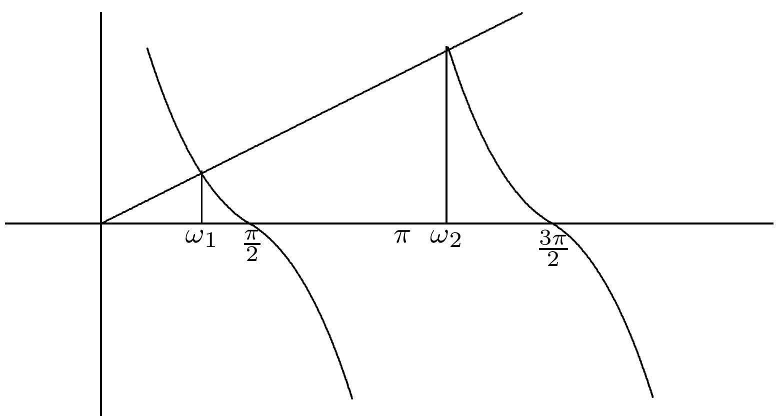
and

$$X'(1) = -2X(1) \quad \text{implies} \quad -\omega \sin\omega = -2\cos\omega.$$

This equation can be written

$$\frac{\omega}{2} = \cot\omega$$

and solved graphically:



The smallest ω is somewhere less than $\frac{\pi}{2}$, and the large ones are slightly greater than $(n-1)\pi$. Negative ω give no new eigenvalues (ω^2).

We should check (extra credit!) that there are no negative eigenvalues (that is, $\omega = i\kappa$, κ real). This follows from the general Sturm–Liouville theorem (or an integration by parts) because the *inward* normal derivative is a positive multiple of X itself. But let's show it directly:

$$X'' = +\kappa^2 X \;\Rightarrow\; X(x) = \cosh(\kappa x) \;\Rightarrow\; \kappa \sinh\kappa = -2\cosh\kappa \;\Rightarrow\; \coth k = -\frac{\kappa}{2}.$$

Since $\coth\kappa$ and $-\kappa$ always have opposite signs, there are no roots. (Also, it's easy to see that 0 is not an eigenvalue.)

Define

$$X_n(x) = \cos(\omega_n x), \qquad \phi_n(x) = \frac{X_n(x)}{\|X_n\|}\,,$$

where

$$\|X_n\|^2 = \int_0^1 X_n(x)^2\,dx.$$

The integral can be evaluated (more extra credit!):

$$\|X_x\|^2 = \int_0^1 \cos^2(\omega_n x)\,dx = \frac{1}{2}\int_0^1 [1+\cos(2\omega_n x)]\,dx = \frac{1}{2} + \frac{1}{4\omega_n}[\cos(2\omega_n) - 1].$$

In terms of ϕ_n the orthonormality relation is

$$\int_0^1 \phi_n(x)^* \phi_m(x)\,dx = \delta_{nm}$$

and the completeness relation is

$$\sum_{n=1}^{\infty} \phi_n(x)\phi_m(y)^* = \delta(x-y).$$

(The stars can be omitted since the eigenfunctions are real.) In terms of X_n the relations are more complicated.

We can now write the general solution as

$$u(t,x) = \sum_{n=1}^{\infty} c_n e^{-\omega_n{}^2 t} \phi_n(x)$$

and calculate the coefficients as

$$c_n = \int_0^1 \phi_n(x)^* f(x)\,dx.$$

3. *(30 pts.)* *Assume* that you have a solution $v(x,y)$ for the *next* problem. (You *don't* need to do that problem before doing this one.) Solve (by separation of variables or an equivalent transform technique)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (0 < t < \infty, \quad -\infty < x < \infty, \quad 0 < y < \infty),$$

$$u(0,x,y) = g(x,y), \qquad u(t,x,0) = f(x).$$

Since v is a steady-state solution for this problem, we need only find $w = u - v$, which satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad (0 < t < \infty, \quad -\infty < x < \infty, \quad 0 < y < \infty),$$

$$w(0,x,y) = g(x,y) - v(x,y) \equiv h(x,y), \qquad w(t,x,0) = 0.$$

Take a Fourier transform in x and a sine transform in y :

$$\frac{\partial \hat{w}}{\partial t} = -k^2 \hat{w} - p^2 \hat{w}, \qquad \hat{w}(0,k,p) = \hat{h}(k,p),$$

where

$$\hat{h}(k,p) \equiv \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, e^{-ikx} \sin(py) w(x,y)$$

and $\hat{w}$ is defined similarly. The solution for the transform is

$$\hat{w}(t,k,p) = \hat{h}(k,p) e^{-(k^2+p^2)t},$$

and hence the solution of the problem is

$$u(t,x,y) = v(x,y) + \frac{1}{\pi} \int_{-\infty}^{\infty} dk \int_0^{\infty} dp\, e^{ikx} \sin(py) e^{-(k^2+p^2)t} \hat{h}(k,p).$$

4. *(30 pts.)* Consider the problem

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (-\infty < x < \infty, \quad 0 < y < \infty),$$

$$v(x,0) = f(x) \qquad (-\infty < x < \infty).$$

Solve the problem, leaving the answer in the Green function form, $v = \int Gf$. (It is up to you to fill in the correct variables, limits of integration, etc.) Write out as many details of the solution process as you need to be confident that your solution is correct. (If you have time, write out the remaining details.)

This one is straight out of the class notes (although not stressed in lecture this year), so I will not type it out. Suffice it to say that the Green function is given by the second famous integral.