Math. 412 (Fulling) 22 September 2006

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

1. *(35 pts.)* Consider the Fourier sine series (over the basic interval $[0, \pi]$) of the function

$$h(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ 0 & \text{for } 1 < x < \pi. \end{cases}$$

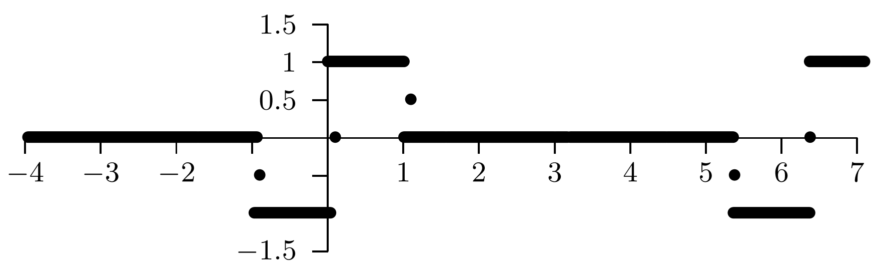
(a) With or without actually calculating them, describe how the coefficients b_n in the series will behave for large n .

This is a step function, so from previous experience we expect the coefficients to fall off like $\frac{1}{n}$. To verify, calculate

$$b_n = \frac{2}{\pi} \int_0^\pi \sin(nx) h(x)\, dx = \frac{2}{\pi} \int_0^1 \sin(nx)\, dx = -\frac{2}{\pi n} \cos(nx)\Big|_0^1 = \frac{2(1 - \cos n)}{\pi n} .$$

(b) Over the interval $[-4, 7]$, sketch the function to which the series converges. On the same axes, sketch what you imagine the 10th partial sum of the series looks like. (Don't try to calculate it.)

The limit function is the odd periodic extension of the original function. At the discontinuities the limit is the average of the left and right limits.



(The graph of the partial sum will be posted in a separate file if I ever get around to it.)

(c) Does the series converge

(i) uniformly?

No, because h is discontinuous.

(ii) pointwise?

Yes, because h is piecewise smooth.

(iii) in the mean?

Yes, because h is square-integrable.

2. *(35 pts.)* Solve the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < L, \quad -\infty < t < \infty),$$

$$u(0,t) = 0 = u(L,t) \qquad (-\infty < t < \infty),$$

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = g(x) \qquad (0 < x < L)$$

by the Fourier method (separation of variables).

Separation step: Look for solutions of the form $X(x)T(t)$. By the usual argument we arrive at

$$T'' = -k^2 T, \qquad X'' = -k^2 X.$$

The boundary conditions $X(0) = 0 = X(L)$ dictate the solutions

$$X_n(x) = \sin\frac{n\pi x}{L}, \quad k_n = \frac{n\pi}{L}, \quad T_n(t) = \sin(k_n t) \text{ or } \cos(k_n t).$$

Superposition step: Try

$$u(x,t) = \sum_{n=1}^{\infty} [a_n \sin(k_n x)\cos(k_n t) + b_n \sin(k_n x)\sin(k_n t)].$$

Impose the initial conditions:

$$0 = \sum_{n=1}^{\infty} a_n \sin(k_n x), \qquad g(x) = \sum_{n=1}^{\infty} k_n b_n \sin(k_n x).$$

Thus $a_n = 0$ and

$$k_n b_n = \frac{2}{L}\int_0^L \sin(k_n x) g(x)\, dx\,;$$

that is,

$$b_n = \frac{2}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\, g(x)\, dx.$$

3. *(30 pts.)* Consider the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty, \quad -\infty < t < \infty),$$

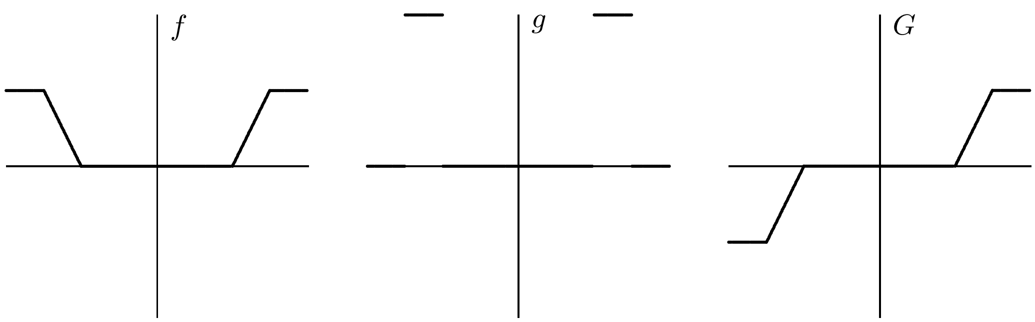
$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad (-\infty < t < \infty),$$

$$u(x,0) = f(x) \equiv \begin{cases} 0 & \text{if } 0 < x \le 1, \\ 2(x-1) & \text{if } 1 < x < 1.5, \\ 1 & \text{if } 1.5 \le x < \infty, \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = f'(x) \qquad (0 < x < \infty).$$

Sketch the solution as a function of x for $t = \frac{1}{2}$, $t = 1$, $t = \frac{3}{2}$, and $t = 2$. (*Hint:* The initial data have been cunningly chosen so that initially the wave moves only in one direction.) Fill in as much or as little d'Alembert theory as you need to to arrive at your sketches.

Because of the Neumann condition, the solution is constructed out of the even extensions of both data functions. This is slightly tricky, because the even extension of f' is not the derivative of [the evenly extended] f when x is negative; instead, it equals $-f'$ there. The function just described is the one we usually call g; its indefinite integral, G, equals f when x is positive and $-f$ when x is negative. Sketches are much easier to work with than formulas in this situation, so let's draw some.



Now recall (or rederive) that

$$u(x,t) = B(x-t) + C(x+t)$$

where

$$B(z) = \tfrac{1}{2}[f(z) - G(z)], \quad C(z) = \tfrac{1}{2}[f(z) + G(z)].$$

It follows that in the present case

$$B(z) = \begin{cases} 0 & \text{for } z > 0, \\ f(z) & \text{for } z < 0; \end{cases} \qquad C(z) = \begin{cases} f(z) & \text{for } z > 0, \\ 0 & \text{for } z < 0. \end{cases}$$

The B term is a right-moving wave that does not emerge into the physical region until after $t = 1$. The C term is a left-moving wave that moves quickly to the boundary so that after $t = 1.5$ it equals

1 everywhere in the physical region. For each fixed t we need to add the two patterns with the proper horizontal shifts: