Math. 412 (Fulling) 20 October 2006

Test B – Solutions

Calculators may be used for simple arithmetic operations only!

Famous integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\,\frac{y}{x^2+y^2}\,.$$

1. *(30 pts.)* Consider Laplace's equation in a quadrant,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < \infty, \quad 0 < y < \infty)\,,$$

with boundary conditions

$$u(0,y) = f(y)\,, \quad u(x,0) = 0\,.$$

Solve the problem by separation of variables or an equivalent transform technique. (Require the solution to be bounded.)

Quick way: Take a Fourier sine transform with respect to y. Let's denote such transforms by a tilde.

$$\frac{\partial^2 \tilde u}{\partial x^2} - k^2 \tilde u = 0, \quad \tilde u(0,k) = \tilde f(k)\,.$$

Since the solution must be bounded as $x \to +\infty$,

$$\tilde u(x,k) = C(k) e^{-kx}\,, \quad \text{and} \quad C(k) = \tilde f(k)\,.$$

So

$$u(x,y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde f(k) e^{-kx} \sin(ky)\, dk$$

if

$$\tilde f(k) \equiv \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \sin(ky)\, dy\,.$$

Outline of long way: Separate variables as

$$u_{\text{sep}} = X(x)Y(y) \;\Rightarrow\; \frac{X'}{X} = -\frac{Y'}{Y} = +k^2\,.$$

Then $Y(0) = 0 \;\Rightarrow\; Y_k(y) = \sin(ky)$ and the boundedness requirement vindicates the choice of sign for k^2. Boundedness in the other dimension forces $X(x) = e^{-kx}$. Now superpose solutions:

$$u(x,y) = \int_0^\infty B(k) e^{-kx} \sin(ky)\, dk$$

with

$$f(y) = \int_0^\infty B(k) \sin(ky)\, dk\,.$$

It follows that

$$B(k) = \frac{2}{\pi} \int_0^\infty f(y) \sin(ky)\, dy\,.$$

2. *(25 pts.)* Do **ONE** of the following ((A) **OR** (B)). *(At most 15 points extra credit for doing both.)*

   (A) Reexpress the solution to Question 1 in Green-function form. (Evaluate the integral giving the Green function.)

Substitute the coefficient formula (for C or B) into the other formula with a change of integration letter, getting

$$\begin{aligned} u(x,y) &= \frac{2}{\pi}\int_0^\infty dk \int_0^\infty dz\, f(z)\sin(kz)e^{-kx}\sin(ky) \\ &= \int_0^\infty G(x,y,z)f(z)\,dz \end{aligned}$$

with

$$\begin{aligned} G(x,y,z) &= \frac{2}{\pi}\int_0^\infty e^{-kx}\sin(ky)\sin(kz)\,dk \\ &= -\frac{1}{2\pi}\int_0^\infty e^{-kx}\left[e^{ik(y+z)} + e^{-ik(y+z)} - e^{ik(y-z)} - e^{ik(z-y)}\right]dk \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{-|k|x}\left[e^{ik(y-z)} - e^{ik(y+z)}\right]. \end{aligned}$$

By the second famous integral, this is

$$\frac{1}{\pi}\left[\frac{x}{x^2+(y-z)^2} - \frac{x}{x^2+(y+z)^2}\right].$$

Quicker way: The Green function for Laplace's equation in the right half-plane is given by the second famous integral with the roles of x and y interchanged. More precisely, it is

$$G_0 = \frac{1}{\pi}\,\frac{x}{x^2+(y-z)^2}\,.$$

By the method of images, the Green function for the quadrant with Dirichlet condition on the horizontal axis is the G we found above.

   (B) Construct a Green function to solve the ODE problem

$$\frac{d^2y}{dx^2} - y = f(x) \qquad (0 < x < 2)\,,$$

$$y(0) = 0\,, \quad y(2) = 0.$$

Hint: $\sinh a\cosh b - \cosh a\sinh b = \sinh(a-b)$.

We need

$$\frac{\partial^2}{\partial x^2}G(x,z) - G(x,z) = \delta(x-z)\,, \qquad G(0,z) = 0 = G(2,z)\,.$$

Therefore, to satisfy the boundary conditions we have

$$G(x,z) = \begin{cases} A\sinh x & \text{for } x < z\,, \\ B\sinh(x-2) & \text{for } x > z\,. \end{cases}$$

Continuity requires

$$A\sinh z = B\sinh(z-2)\,.$$

The jump condition is (for $\epsilon \downarrow 0$)

$$\frac{\partial}{\partial x}G(z+\epsilon,z) - \frac{\partial}{\partial x}G(z-\epsilon,z) = 1 \;\Rightarrow\; B\cosh(z-2) - A\cosh z = 1\,.$$

Solve:

$$A = \frac{\begin{vmatrix} 0 & -\sinh(z-2) \\ 1 & \cosh(z-2) \end{vmatrix}}{\begin{vmatrix} \sinh z & -\sinh(z-2) \\ -\cosh z & \cosh(z-2) \end{vmatrix}}\,, \qquad B = \frac{\begin{vmatrix} \sinh z & 0 \\ -\cosh z & 1 \end{vmatrix}}{\begin{vmatrix} \sinh z & -\sinh(z-2) \\ -\cosh z & \cosh(z-2) \end{vmatrix}}\,.$$

Do the algebra:

$$A = \frac{\sinh(z-2)}{\sinh 2}\,, \qquad B = \frac{\sinh z}{\sinh 2}\,.$$

The answer can be summarized by

$$G(x,z) = \frac{\sinh(x_<)\sinh(x_> - 2)}{\sinh 2}\,, \qquad \text{where } x_< \equiv \min(x,z), \quad x_> \equiv \max(x,z)\,.$$

3. *(Essay – 45 pts.)* Outline a strategy to solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad (0 < x < \pi\,, \quad 0 < y < \pi\,, \quad 0 < t < \infty)\,,$$

$$u(x,0,t) = 0\,, \quad u(x,\pi,t) = g(x)\,,$$

$$u(0,y,t) = T = u(\pi,y,t) \qquad (T = \text{nonzero constant})\,,$$

$$u(x,y,0) = f(x,y)\,.$$

Then carry out as many of the steps as you have time for.

Because there are three different kinds of nonhomogeneous data, we expect to have to break the solution into three terms. Because the boundary data are independent of t, we expect that two of those terms can be steady-state solutions, independent of t. Suppose that we have found a steady-state solution, $v(x,y)$, that satisfies all the spatial boundary conditions. (Later we will break v into two parts.) Then $w = u - v$ satisfies the heat equation with completely homogeneous boundary conditions and the initial condition $w(x,y,0) = f(x,y) - v(x,y)$. The problem for w can be solved by separation of variables, and the eigenfunctions will be products of sine functions in the x and y directions.

Now consider the problem of finding v. It will satisfy Laplace's equation. Since T does not depend on y, the best strategy is to look for a solution of Laplace's equation that does not depend on y and satisfies the T boundary conditions. It is fairly easy to see that the constant function T will work, so I won't bother to give that function another name. Let $s(x,y) = v(x,y) - T$. Then s must satisfy Laplace, vanish on the sides $x = 0, \pi$, and satisfy

$$s(x,0) = -T\,, \quad s(x,\pi) = g(x) - T\,.$$

This can be solved by a routine separation of variables. (There will be two kinds of terms corresponding to the two nonhomogeneous boundary data functions, but since both those conditions refer to the same variable (i.e., parallel sides) it is not necessary to split s into two parts before separating variables.)

Finally, the solution is $u = w(x,y,t) + s(x,y) + T$. [An alternative, but suboptimal, solution is to write $v = v_1 + v_2$ with g as boundary data for v_1 and T as boundary data as v_2 (and no other nonhomogeneous data). In that approach one will need to expand the constant function T as a sine series in x. (But we'll end up doing that anyway, it turns out.)]

Sketch of details of w*:* It satisfies

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\,,$$

$$w(x,0,t) = 0 = w(x,\pi,t)\,, \qquad w(0,y,t) = 0 = w(\pi,y,t)\,,$$

$$w(x,y,0) = f(x,y) - v(x,y) \equiv h(x,y)\,.$$

Find eigenfunctions $X_n(x)Y_m(y) = \sin(nx)\sin(my)$. Then

$$w(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\sin(nx)\sin(my)e^{-(n^2+m^2)t}$$

and

$$a_{nm} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} \sin(nx)\sin(my)h(x,y)\,dx\,dy\,.$$

Sketch of details of T*:* We need a function of x alone that satisfies Laplace's equation (i.e., has zero second derivative) and equals T at two points. It is of the form $Ax + B$, and you quickly find $A = 0$, $B = T$.

Sketch of details of s*:* It satisfies

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} = 0\,,$$

$$s(x,0) = -T\,, \quad s(x,\pi) = g(x) - T \equiv k(x)\,,$$

$$s(0,y) = 0 = s(\pi,y)\,.$$

Separation of variables leads to eigenfunctions $X_n(x) = \sin(nx)$ and complementary solutions $Y(y) = \sinh(ny)$ and $\sinh[n(\pi - y)]$ (each chosen to vanish on one of the relevant boundaries). Summing up,

$$s(x,y) = \sum_{n=1}^{\infty} \sin(nx)\Big[a_n\sinh(ny) + b_n\sinh[n(\pi - y)]\Big]\,.$$

Then we find a_n from the Fourier sine coefficients of k and b_n from those of $-T$ (divided by $\sinh(n\pi)$).