Math. 412 (Fulling) 17 November 2006

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

Useful information:
Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

1. *(60 pts.)* Consider the eigenvalue problem

$$\phi'' = -\lambda\phi \quad (0 < x < 1)\,; \qquad \phi'(0) = \beta\phi(0)\,, \quad \phi(1) = 0\,.$$

NOTE: You should be able to answer (c) and (d) even if you don't have a clue about (a) and (b).

(a) Assuming for now that the eigenvalues λ are all positive, find the eigenfunctions and show how to find the eigenvalues approximately (by a graphical method) if $\beta = 2$.

The easiest way is to write $\phi(x) = \sin[\omega(1-x)]$ (where $\omega^2 = \lambda$) to satisfy the second boundary condition, and then to calculate from the other boundary condition that $-\omega\cos\omega = \beta\sin\omega$. Another way is to write $\phi(x) = A\cos(\omega x) + B\sin(\omega x)$ with A and B unknown. The two boundary conditions give two homogeneous linear equations for A and B. Setting the determinant of the system equal to 0 yields again $-\omega\cos\omega = \beta\sin\omega$. Then you can solve for A and B up to a constant, getting something equivalent to $C\sin[\omega(1-x)]$ after a trig identity.

In either case, you then solve $-\dfrac{\omega}{\beta} = \tan\omega$ with $\beta = 2$ graphically as in Fig. 5.8.1 of Haberman.

(b) Show that the eigenvalues are indeed all positive if $\beta > 0$. Describe how an eigenvalue might be negative if $\beta < 0$.

Approach 1: Observe that

$$\lambda \int_0^1 |\phi(x)|^2\,dx \equiv \lambda\langle\phi,\phi\rangle = \langle\phi,(-\phi'')\rangle = -\int_0^1 \phi(x)^*\phi''(x)\,dx$$

$$= -\phi(1)^*\phi'(1) + \phi(0)^*\phi'(0) + \int_0^1 |\phi'(x)|^2\,dx = 0 + \beta|\phi(0)|^2 + \int_0^1 |\phi'(x)|^2\,dx.$$

Divide by $\langle\phi,\phi\rangle$ ("Rayleigh quotient") to see that λ is a sum of obviously positive quantities, provided that $\beta > 0$. If $\beta < 0$ we can't draw that conclusion, but to see clearly whether there *must* be a negative eigenvalue in that case, we need to resort to the second approach.

Approach 2: Suppose that $\lambda = -\rho^2$. Then the solution of the ODE is $\phi(x) = \sinh[\rho(1-x)]$, and a calculation analogous to the one in (a) leads to

$$\tanh \rho = -\frac{\rho}{\beta} .$$

This has a (nontrivial) solution if and only if $-1 < \beta < 0$, as shown by drawing a graph like Fig. 5.8.3. (For negative β there are still infinitely many positive eigenvalues, as shown by Figs. 5.8.2 and 5.8.4.)

(c) Write out the orthogonality and completeness relations obeyed by the eigenfunctions. Given the eigenfunctions $\phi_n(x) = \sin[\omega_n(1-x)]$ from (a) (or just the notation ϕ_n !) you can define orthonormal eigenfunctions

$$\psi_n(x) = \frac{\phi_n(x)}{\|\phi_n\|}, \quad \|\phi_n\|^2 \equiv \int_0^1 |\phi_n(x)|^2 \, dx .$$

(The integral can be evaluated, but it is *not* the usual $\pi/2$ or $L/2$ of ordinary Fourier series.) Then the orthogonality and completeness relations are

$$\int_0^1 \psi_n(x)^* \psi_m(x) \, dx = \delta_{mn} , \quad \sum_{n=1}^{\infty} \psi_n(x)\psi_n(y)^* = \delta(x-y) .$$

(In this case the complex conjugates are superfluous, since we chose the eigenfunctions real. If there is a negative eigenvalue, you need to include a term for its eigenfunction in the completeness relation (but I should have excluded that case in the problem statement).)

You can also write the equivalent relations in terms of the ϕ_n, including the necessary factors of $\|\phi_n\|$.

(d) Use the (normalized) eigenfunctions and eigenvalues to solve the heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < 1, \ t > 0),$$

$$\frac{\partial u}{\partial x}(0,t) = \beta u(0,t), \quad u(1,t) = 0, \qquad u(x,0) = f(x).$$

I'll use the normalized functions ψ_n to keep the formulas simple. In the usual way you get (with $\lambda_n = {\omega_n}^2$, still assuming $\beta > 0$)

$$u(x,t) = \sum_{n=1}^{\infty} C_n \psi_n(x) e^{-\lambda_n t} .$$

Then

$$f(x) = \sum_{n=1}^{\infty} C_n \psi_n(x) ,$$

so

$$C_n = \langle \psi_n, f \rangle = \int_0^1 \psi_n(x)^* f(x) \, dx .$$

If you use unnormalized eigenfunctions, you get $\int_0^1 |\phi_n(x)|^2 \, dx$ in the denominator (without the square root, since you have redefined C_n).

2. *(40 pts.)* Do **ONE** of the following ((A) **OR** (B)). (*Sorry, no extra credit for doing both.* **Clearly indicate which question you want graded.**)

(A) Solve Laplace's equation in a sector,

$$\nabla^2 u = 0 \quad \left(0 < r < 3\,,\ 0 < \theta < \frac{\pi}{3}\right),$$

$$\frac{\partial u}{\partial \theta}(r,0) = 0 = \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{3}\right), \qquad u(3,\theta) = g(\theta)\,.$$

Writing $u = R\Theta$ yields

$$\Theta'' = -\mu^2 \Theta\,, \quad R'' + \frac{1}{r}R' - \frac{\mu^2}{r^2}R = 0\,.$$

From the homogeneous boundary conditions we get $\Theta(\theta) = \cos(3m\theta)$, $\mu = 3m$, $m = 0, 1, \ldots$. ($L = \pi/3 \ \Rightarrow\ m\pi/L = 3m$, $2/L = 6/\pi$.) The two independent solutions of the R equation are $r^{\pm 3m}$, and possibly $\ln r$ in the case $m = 0$, but for regularity at $r = 0$ we choose the nonnegative exponents (and no log). Thus

$$u(r,\theta) = \sum_{m=0}^{\infty} a_m r^{3m} \cos(3m\theta)\,.$$

Then

$$g(\theta) = \sum_{m=0}^{\infty} a_m 3^{3m} \cos(3m\theta)\,.$$

So if $m \neq 0$,

$$a_m = 3^{-3m}\,\frac{6}{\pi}\int_0^{\pi/3} \cos(3m\theta) g(\theta)\, d\theta,$$

$$a_0 = \frac{3}{\pi}\int_0^{\pi/3} g(\theta)\, d\theta\,.$$

(B) Solve the wave equation in a disk,

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2} \quad (0 \le r < 3\,,\ 0 \le \theta < 2\pi\,,\ 0 < t < \infty)\,,$$

$$u(3,\theta,t) = 0\,, \qquad u(r,\theta,0) = f(r,\theta)\,, \quad \frac{\partial u}{\partial t}(r,\theta,0) = 0\,.$$

[This is a special case of the "drum problem" done in the class notes, pp. 106–107 and 112–115.]