Asymptotics of Bessel Functions

We were naturally led to Bessel's equation in the generalized form

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\omega^2 - \frac{n^2}{r^2}\right)R = 0.$$

Its solutions are called $J_n(\omega r)$ and $Y_n(\omega r)$. Our aim now is to gain some understanding of how the previously stated formulas for the approximate behavior of the Bessel functions in the limit of large r are derived.

Notice that the limit of large r at fixed positive ω is equivalent to the limit of large ω at fixed positive r. So we proceed, assuming that $\omega r \gg n$. Let's search for a solution of the form $R = Ae^{i\omega S}$, where S and A are functions of r; A may also depend on ω but stays bounded as $\omega \to \infty$.

$$R' = i\omega S' A e^{i\omega S} + A' e^{i\omega S},$$

$$R'' = -\omega^2 (S')^2 A e^{i\omega S} + i\omega S'' A e^{i\omega S} + 2i\omega S' A' e^{i\omega S} + A'' e^{i\omega S}.$$

Therefore, the equation is equivalent to

$$0 = -\omega^2\left[(S')^2 - 1\right]A + i\omega\left(S''A + 2S'A' + \frac{S'A}{r}\right) + A'' + \frac{A'}{r} - \frac{n^2}{r^2}A. \qquad (*)$$

The only way $(*)$ can be true, given our assumption about the behavior of A, is to have

$$(S')^2 = 1.$$

Thus $S' = \pm 1$, so

$$S = \pm r \quad (+\,\text{constant}).$$

The two signs will yield two linearly independent solutions; the constant contributes a trivial constant phase factor, so it can be ignored.

After division by $i\omega$ the rest of $(*)$ is (since $S'' = 0$)

$$0 = \pm 2A' \pm \frac{A}{r} - \frac{i}{\omega}\left(A'' + \frac{A'}{r} - \frac{n^2 A}{r^2}\right).$$

Now assume that A has an expansion of the form

$$A \sim \sum_{j=0}^{\infty} \omega^{-j} A_j\,.$$

(I write "$\sim$" instead of "$=$" because the series may not converge. All that is claimed is that the first few terms of the series are a good approximation when ω is sufficiently small. (And today we won't *prove* even that.)) Well, we get

$$A' \sim \sum_{j=0}^{\infty} \omega^{-j} A'_j \,, \qquad A'' \sim \sum_{j=0}^{\infty} \omega^{-j} A''_j \,,$$

and hence

$$0 = \pm 2 \sum_{j=0}^{\infty} \omega^{-j} A'_j \pm \sum_{j=0}^{\infty} \omega^{-j} \frac{A_j}{r} - i \sum_{j=0}^{\infty} \omega^{-(j+1)} \left(A''_j + \frac{A'_j}{r} - \frac{n^2 A_j}{r^2} \right). \tag{$\dagger$}$$

The last term in ($\dagger$) can be rewritten

$$-i \sum_{j=1}^{\infty} \omega^{-j} \left(A''_{j-1} + \frac{A'_{j-1}}{r} - \frac{n^2 A_{j-1}}{r^2} \right).$$

We can now set the expression multiplying each ω^{-j} successively to 0. For $j = 0$ we get

$$0 = \pm 2 A'_0 \pm \frac{A_0}{r} \,,$$

hence $A'_0 = -A_0/(2r)$, a separable equation whose solutions are

$$A_0 = C r^{-1/2}.$$

So far we have

$$R(r) \sim A_0 e^{i\omega S} = C \, \frac{e^{\pm i\omega r}}{\sqrt{r}} \,.$$

This is exactly what we expected! Recall that according to the handbooks there are two special solutions of Bessel's equation,

$$H_n^{(1)}(\omega r) \sim \sqrt{\frac{2}{\pi \omega r}} (-i)^{n+\frac{1}{2}} e^{i\omega r},$$
$$H_n^{(2)}(\omega r) \sim \sqrt{\frac{2}{\pi \omega r}} i^{n+\frac{1}{2}} e^{-i\omega r},$$

and J is the sum of these,

$$J_n(\omega r) \sim \sqrt{\frac{2}{\pi \omega r}} \cos\left(\omega r - \frac{n\pi}{2} - \frac{\pi}{2} \right). \tag{$\ddagger$}$$

Let's press on to the $j = 1$ term in ($\dagger$). Setting $C = 1$, we get

$$\begin{aligned} 2A'_1 + \frac{A_1}{r} &= \mp i \left(A''_0 + \frac{A'_0}{r} - \frac{n^2 A_0}{r^2} \right) \\ &= \mp i \left(\frac{1}{4} - n^2 \right) r^{-5/2}. \end{aligned}$$

This is like the $j = 0$ equation but with a nonhomogeneous term, so we can solve it by using the reciprocal of A_0 as an integrating factor. That is, multiply by $r^{+1/2}$ to get

$$\frac{d}{dr}(2r^{1/2}A_1) = 2A_1' r^{1/2} + A_1 r^{-1/2} = \pm i \left(n^2 - \frac{1}{4}\right) r^{-2}.$$

The solution is

$$A_1 = \mp i \left(n^2 - \frac{1}{4}\right) \frac{1}{2r^{3/2}} + \frac{C}{\sqrt{r}}\,.$$

We can set $C = 0$ because that term can be absorbed into A_0 ; with this understanding, the additional factor of $1/\omega$ in $\omega^{-1}A_1$ is accompanied by an additional factor of r (relative to the A_0 term), in keeping with our expectation that ω and r should always appear together.

In summary,

$$R(\omega r) \sim \frac{e^{\pm i\omega r}}{\sqrt{r}} \left[1 \mp \frac{i}{2\omega r}\left(n^2 - \frac{1}{4}\right) + \cdots\right].$$

We could continue and derive higher-order terms.

That was the easy part. The harder mystery is why J_n , defined as the solution with a certain behavior as $r \to 0$, is the solution with *particular* large-r behavior (‡). Both the phase shift inside the cosine and the normalization factor $\sqrt{2/\pi}$ deserve to be explained.

Unfortunately, we do not have either the time or the mathematical tools (from complex analysis) to answer these questions. Some inkling of what is involved can be obtained from the simpler example of the Airy function, the subject of my next bulletin.