Math. 412 11 December 2007

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Some possibly useful information

Laplacian operator in spherical coordinates ($\theta =$ polar angle, $\phi =$ azimuthal angle):

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\,.$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi).$$

Legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } \ P_l(\cos\theta)\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } \ J_n(z) \ \text{ and } \ Y_n(z)\,.$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{l(l+1)}{z^2}\right) Z = 0 \quad \text{has solutions } \ j_l(z) \ \text{ and } \ y_l(z)\,.$$

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1. *(30 pts.)* Classify each of these equations as linear homogeneous, linear nonhomogeneous, or nonlinear; **also**, classify it as elliptic, parabolic, or hyperbolic.

(a) $\nabla^2 u = -\omega^2 u \quad (\,\omega = \text{constant}\,)\,.$

Linear homogeneous, elliptic.

(b) $\dfrac{\partial^2 u}{\partial x\,\partial y} = u^2\,.$

Nonlinear, hyperbolic. (The second-derivative terms transform to $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial z^2}$.)

(c) $\dfrac{\partial u}{\partial s} = \dfrac{\partial^2 u}{\partial z^2} + 2\,\dfrac{\partial u}{\partial z} + (s^2 + z^2)u\,.$

Linear homogeneous, parabolic.

2. *(Essay – 10 pts.)* Pick one of the three classes — elliptic, parabolic, or hyperbolic — and describe some of the properties specific to solutions of equations of that type.

[See the last section of the class notes, or the solutions to the Fall 2000 final exam (question 6). Note that the maximum principle in its strong form does not apply without additional conditions on the equation.]

3. *(40 pts.)* Solve Laplace's equation in a rectangle,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < L\,,\ 0 < y < \pi)\,,$$

$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \frac{\partial u}{\partial y}(x,\pi) = f(x)\,, \quad u(0,y) = g(y)\,, \quad u(L,y) = 0\,.$$

Let's write the solution as $u = u_1 + u_2$, where

$$\frac{\partial u_1}{\partial y}(x,\pi) = f(x)\,, \quad u_1(0,y) = 0\,, \qquad \frac{\partial u_2}{\partial y}(x,\pi) = 0\,, \quad u_2(0,y) = g(y)\,,$$

and all the other homogeneous conditions remain the same.

Separating variables for u_1 leads to modes of the type $\sin(n\pi x/L)Y(y)$ where $Y'' = +(n\pi/L)^2 Y$ and $Y'(0) = 0$, so $Y(y) \propto \cosh(n\pi y/L)$. So

$$u_1(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right).$$

We need

$$f(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi^2}{L}\right).$$

Therefore,

$$b_n = \frac{2}{L}\,\frac{L}{n\pi \sinh\left(\frac{n\pi^2}{L}\right)} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)\,dx\,.$$

For the other problem we have modes of the type $\cos(ny)X(x)$ where $X'' = +n^2 X$ and $X(L) = 0$. Thus $X(x) \propto \sinh\big(n(L-x)\big)$ if $n \neq 0$, and $X(x) \propto L - x$ if $n = 0$. Therefore,

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \cos(ny) \sinh\big(n(L-x)\big) + a_0(L-x).$$

We need

$$g(y) = \sum_{n=1}^{\infty} a_n \cos(ny) \sinh(nL) + a_0 L.$$

Therefore,

$$a_0 = \frac{1}{\pi L} \int_0^{\pi} g(y)\,dy$$

and for other n

$$a_n = \frac{2}{\pi \sinh(nL)} \int_0^{\pi} \cos(ny) g(y)\,dy.$$

4. *(40 pts.)* Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L)\,, \qquad u(0,t) = 0\,, \quad \frac{\partial u}{\partial x}(L,t) = -\beta u(L,t).$$

(a) Solve the problem with initial data $u(x,0) = f(x)$. Assume that $\beta > 0$ and that all the eigenvalues that arise are positive.

After extracting the time dependence we have the eigenvalue problem

$$X'' = -\omega^2 X\,, \qquad X(0) = 0\,, \quad X'(L) = -\beta X(L)\,.$$

The solutions must be of the form $X_n(x) = \sin(\omega_n x)$ with $\omega_n \cos(\omega_n L) = -\beta \sin(\omega_n L)$. The eigenvalue condition is best written as

$$\tan(\omega_n L) = -\frac{\omega_n}{\beta}\,.$$

Sketch both sides of the equation and pick out the intersections (see Haberman, Fig. 5.8.1).

So, assuming the numbers ω_n known, we construct the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) e^{-\omega_n{}^2 t}$$

and calculate the coefficients

$$c_n = \frac{\int_0^L X_n(x) f(x)\,dx}{\int_0^L X_n(x)^2\,dx}\,.$$

The normalization integral can be evaluated:

$$\int_0^L \sin(\omega_n x)^2\,dx = \frac{1}{2}\int_0^L [1 - \cos(2\omega_n x)]\,dx = \frac{L}{2} - \frac{1}{4\omega_n}\sin(2\omega_n L)\,.$$

(b) Prove that if $\beta > 0$, then all the eigenvalues are indeed positive.

There are several good arguments.

Proof 1: Since $0 < \omega_1 L < \pi$, $X_1(x) = \sin(\omega_1 x)$ has no nodes inside the interval. (*Contrast* Haberman Fig. 5.4.4 for the case $\beta < -1/L$.) Therefore, by the general Sturm–Liouville theorem, $\omega_1{}^2$ is the smallest eigenvalue.

Proof 2 (brute force): Let's look for a negative eigenvalue: Try to solve

$$X'' = +\kappa^2 X\,, \qquad X(0) = 0\,, \quad X'(L) = -\beta X(L)\,.$$

We must have $X_n(x) = \sinh(\kappa_n x)$ with

$$\tanh(\kappa_n L) = -\frac{\kappa_n}{\beta}\,.$$

It is easy to see that the latter equation has no nonzero solutions. Similarly, one can show that 0 is not an eigenvalue.

Proof 3 (Rayleigh quotient): Multiply the ODE, $X'' = -\lambda X$, by X and integrate:

$$\int_0^L X(x)X''(x)\,dx = -\lambda \int_0^L X(x)^2\,dx\,.$$

Integrate the left-hand side by parts, using the boundary conditions:

$$-\int_0^L X'(x)^2\,dx + X(L)X'(L) - X(0)X'(0) = -\int_0^L X'(x)^2\,dx - \beta X(L)^2\,.$$

We end up with

$$\lambda \int_0^L X(x)^2\,dx = \int_0^L X'(x)^2\,dx + \beta X(L)^2\,,$$

where every factor is obviously positive if β is positive. Therefore, λ must be positive.

5. *(40 pts.)*

(a) Using Fourier series, solve the wave equation on a circle (periodic boundary conditions),

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\pi < x < \pi\,, \qquad u(x,0) = f(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = g(x)\,.$$

The simplest version of the solution uses complex exponentials for the x dependence and trig functions for the t dependence:

$$u(x,t) = \sum_{n=-\infty}^{\infty} e^{inx}[a_n \cos(nt) + b_n \sin(nt)]\,.$$

There is one complication: When $n = 0$, $\sin(nt)$ must be replaced by t. (We ran into a similar situation in Qu. 3 above.) Now

$$f(x) = \sum_{n=-\infty}^{\infty} e^{inx} a_n\,, \qquad g(x) = \sum_{n\neq 0} e^{inx} n b_n + b_0\,.$$

Thus

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} f(x)\,dx\,, \qquad b_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx\,, \qquad b_n = \frac{1}{2\pi n}\int_{-\pi}^{\pi} e^{-inx} g(x)\,dx\,.$$

(b) Rearrange your solution to show that it has the structure you would expect from d'Alembert's principle. (If you had trouble with (a), solve the problem directly by d'Alembert's method.)

$$u(x,t) = \sum_{n\neq 0} e^{inx}\left[a_n \frac{e^{int}+e^{-int}}{2} + b_n \frac{e^{int}-e^{-int}}{2i}\right] + a_0 + b_0 t$$
$$= \sum_{n\neq 0}[c_n e^{in(x+t)} + d_n e^{in(x-t)}] + \frac{1}{2}(a_0+a_0) + \frac{b_0}{2}[(x+t)-(x-t)]$$

for some coefficients c_n and d_n . This verifies that the solution is a sum of left-moving and right-moving waves. With more effort (not required) one could show that it has precisely the d'Alembert form,

$$u(x,t) = \frac{1}{2}[f(x-t)+f(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} g(z)\,dz$$

where f and g are the 2π-periodic extensions of the original data functions (the sums of their Fourier series constructed in (a)).

6. *(40 pts.)* We will work in the interior of a sphere of radius a , using "physicists' notation" — that is, (r,θ,ϕ) instead of Haberman's (ρ,ϕ,θ) . (Feel free to skip separation-of-variables steps if you know the outcome in advance.)

(a) *Honors:* Solve the heat equation,

$$\nabla^2 u = \frac{\partial u}{\partial t} \quad \text{for} \quad r<a\,, \quad t>0\,, \qquad u(a,\theta,\phi,t)=0, \quad u(r,\theta,\phi,0) = G(r,\theta,\phi)\,.$$

Use the spherical Bessel function that is regular at the origin.

$$u(r,\theta,\phi,t) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{j=1}^{\infty} c_{lmj}\, j_l(\omega_{lj} r) Y_l^m(\theta,\phi) e^{-\omega_{lm}{}^2 t}\,,$$

where $j_l(\omega_{lj}a)=0$ and

$$c_{lmj} = \frac{\int_0^a r^2\,dr \int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\, j_l(\omega_{lj}r) Y_l^m(\theta,\phi)^* G(r,\theta,\phi)}{\int_0^a r^2\,dr\, j_l(\omega_{lj}r)^2}\,.$$

Regular: Solve Laplace's equation,

$$\nabla^2 u = 0 \quad \text{for} \quad r<a\,, \qquad u(a,\theta,\phi) = F(\theta,\phi)\,.$$

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} r^l Y_l^m(\theta,\phi)$$

where

$$c_{lm} = a^{-l}\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi\, Y_l^m(\theta,\phi)^* F(\theta,\phi)\,.$$

(b) Express your solution to (a) in Green-function form. (Don't expect to be able to evaluate the sums or integrals that arise.)

Regular case: Rewrite

$$c_{lm} = a^{-l} \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\phi'\, Y_l^m(\theta',\phi')^* F(\theta',\phi')\,.$$

Substitute into the solution and pull the integrals outside the sums:

$$u(r,\theta,\phi) = \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\phi'\, G(r,\theta,\phi,\theta',\phi') F(\theta',\phi')$$

with

$$G(r,\theta,\phi,\theta',\phi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{r}{a}\right)^l Y_l^m(\theta,\phi) Y_l^m(\theta',\phi')^*\,.$$

The honors case is similar, but with triple sums and integrals.