Math. 412 (Fulling) 17 September 2007

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. *(50 pts.)* Consider the Fourier sine series (over the basic interval $[0,\pi]$) of the function

$$F(x) = \begin{cases} x & \text{for } 0 \le x \le 1, \\ 1 & \text{for } 1 < x < \pi. \end{cases}$$

(a) Write down the form of the series (with coefficients called b_n).

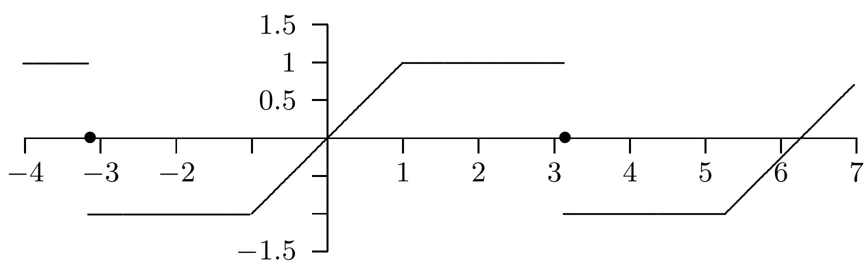
$$F(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

(b) Write down the formula for b_n . (Don't evaluate it yet.)

$$b_n = \frac{2}{\pi}\int_0^\pi \sin(nx)F(x)\,dx\,.$$

(c) Over the interval $[-4,7]$ (i.e., for $-4 \le x \le 7$), sketch the function to which the series converges. On the same axes, sketch what you imagine the 5th partial sum of the series looks like. (Don't try to calculate it.)

The function is the odd periodic extension of F . Its graph includes the points at $x=\pm\pi$ and $y=0$ (equal to the average of the right and left limits of F there).



(The partial sum will be displayed in a *Maple* session later.)

(d) *(regular)* Calculate b_n (using your formula from part (b)).

It is the sum of two integrals:

$$\int_0^1 x\sin nx\,dx = -\,x\,\frac{\cos nx}{n}\bigg|_0^1 + \int_0^1 \frac{\cos nx}{n}\,dx = -\frac{\cos n}{n} + \frac{\sin n}{n^2}\,.$$

$$\int_1^\pi \sin nx\,dx = -\frac{\cos nx}{n}\bigg|_1^\pi = -\frac{(-1)^n}{n} + \frac{\cos n}{n}\,.$$

Thus

$$b_n = \frac{2}{\pi}\left(\frac{(-1)^{n-1}}{n} + \frac{\sin n}{n^2}\right).$$

(d) *(honors)* Does the series converge

(i) uniformly?

No. The jumps at $x = \pm\pi$ can't be reproduced by a uniform limit of continuous functions.

(ii) pointwise?

Yes, F is continuous and piecewise smooth.

(iii) in the mean?

Yes. F is square-integrable.

(e) Use the Fourier series to solve the wave problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \pi, \quad 0 < t < \infty),$$

$$u(0,t) = 0 = u(\pi,t), \qquad u(x,0) = F(x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

(Don't rehearse all the steps of separation of variables; we know that the sine functions in this problem are the right eigenfunctions. Leave answer in terms of b_n — don't substitute from (b) or (d).)

The functions $\sin nx \cos nt$ and $\sin nx \sin nt$ solve the wave equation and the boundary conditions $u(0,t) = 0 = u(\pi,t)$. Because of the initial condition $\frac{\partial u}{\partial t}(x,0) = 0$, only the first of these will appear in our problem. So we expect the solution to have the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin nx \cos nt$$

for some numbers b_n , with

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx\,.$$

Since this is the series in (a), the solution for b_n is the formula in (b).

2. *(50 pts.)* Consider the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad (0 < x < \infty, \quad -\infty < t < \infty),$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad (-\infty < t < \infty),$$

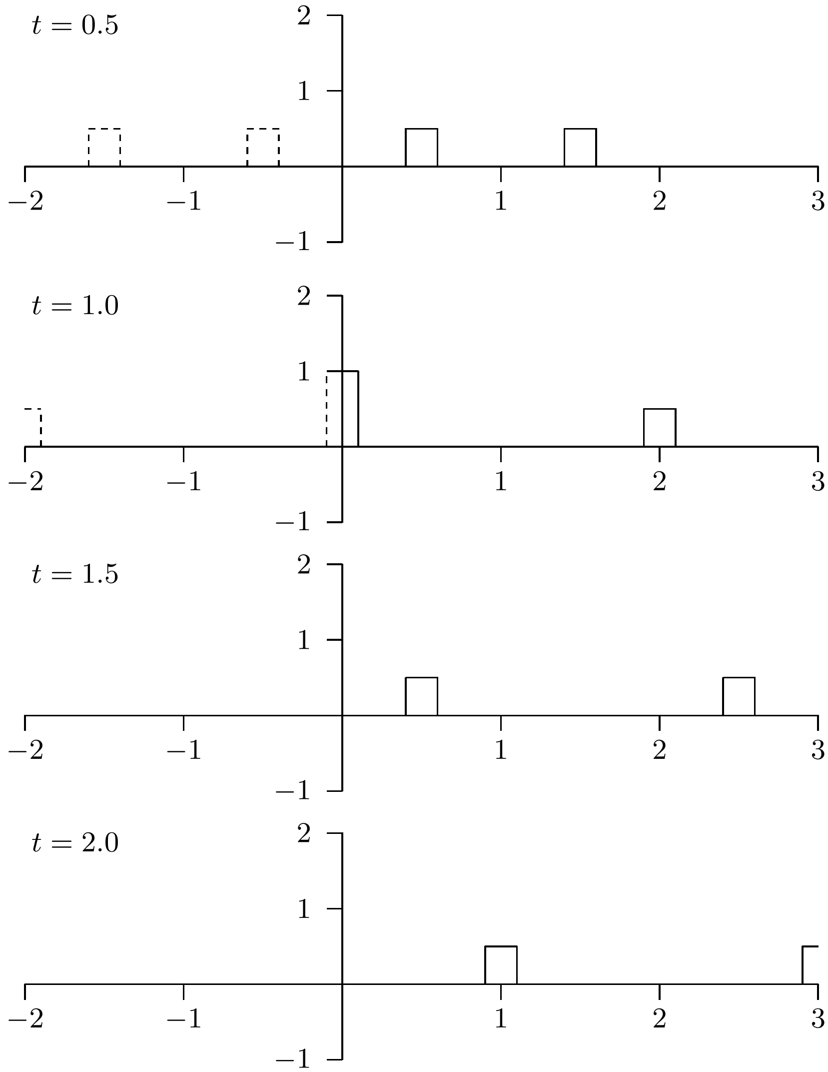
$$u(x,0) = h(x) \equiv \begin{cases} 1 & \text{if } 0.9 < x \le 1.1, \\ 0 & \text{if } 0 < x < 0.9 \text{ or } x > 1.1, \end{cases}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad (0 < x < \infty).$$

Answer (a) and (b) in whichever order you prefer.

(a) Sketch the solution as a function of x for $t = 0.5$, $t = 1$, $t = 1.5$, and $t = 2$.

The initial pulse breaks into two pieces, traveling right and left. When the leftward pulse hits the boundary, it reflects back right-side-up, because of the Neumann boundary condition. We can also think of the reflected pulse as a pulse moving in from the unphysical region ($x < 0$) in the solution in the whole space-time whose initial data function is the even extension of the given data (see (b)).



(b) Write down the solution by the d'Alembert method.

$$u(x,t) = \tfrac{1}{2}[f(x+t) + f(x-t)],$$

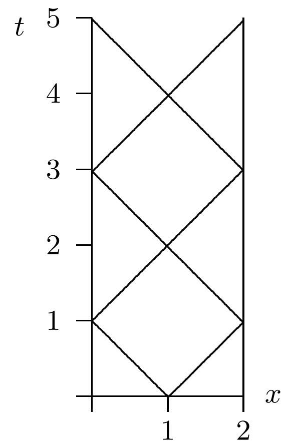
where f is the even extension of h ; that is, $f(x) = h(x)$ for positive x , and

$$f(x) = \begin{cases} 1 & \text{if } -1.1 \le x < -0.9, \\ 0 & \text{for other negative } x. \end{cases}$$

(c) Suppose that another boundary is inserted, so that

$$\frac{\partial u}{\partial x}(2,t) = 0.$$

Sketch the paths in space-time followed by the wave pulses for $0 < x < 2$, $0 < t < 5$.



(d) Describe how the answers to (a) and (b) will change if

(regular) The boundary condition is replaced by

$$u(0,t) = 0 \quad (-\infty < t < \infty).$$

The pulse will reflect upside down. The appropriate extension of the data function is the odd extension, but otherwise (b) is unchanged.

(or) *(honors)* The initial conditions are replaced by

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial t}(x,0) = h(x) \quad \text{(same } h \text{ as above).}$$

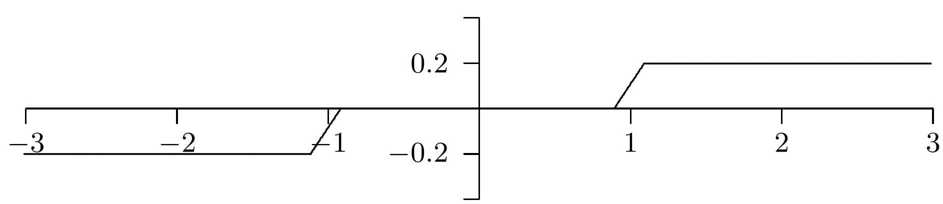
This time we have

$$u(x,t) = \tfrac{1}{2}[G(x+t) - G(x-t)],$$

where

$$G(z) = \int_0^z g(\tilde{z})\, d\tilde{z}$$

and g is the even extension of h. This G is an odd function whose graph is



Again there are left- and right-moving parts of the solution, but now they are "slabs" or "lava flows", not localized pulses. Nevertheless, for a fixed t, the two terms in u will cancel each other when $|x|$ is sufficiently large, so that the signal can't travel from the initially disturbed interval faster than the wave speed ($c = 1$). If you had time to sketch the solution for various times, congratulations.