Math. 412 (Fulling) 16 November 2007

Test C – Solutions

Calculators may be used for simple arithmetic operations only!

Useful information:
Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

1. *(50 pts.)* Solving the heat or wave equation in an annulus (ring-shaped region) would lead to an eigenvalue problem

$$\nabla^2 \Phi = -\omega^2 \Phi \quad (r_1 < r < r_2\,, \quad 0 \le \theta < 2\pi)\,,$$
$$\Phi(r_1, \theta) = 0 = \Phi(r_2, \theta)\,,$$
$$\text{periodic boundary conditions in } \theta\,.$$

In turn, this problem has solutions of the form

$$\Phi_n(r, \theta) = R_{nj}(r)\sin(n\theta)\,, \qquad \Psi_n(r, \theta) = R_{nj}(r)\cos(n\theta)\,.$$

(a) Find the allowed eigenfunctions R_{nj} as explicitly as you can.

Substituting the given form of Φ or Ψ into the equation, we get

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2} R + \omega^2 R = 0\,.$$

Letting $z = \omega r$ scales out ω to reduce this to Bessel's equation. Therefore,

$$R(r) = aJ_n(\omega r) + bY_n(\omega r)\,.$$

The boundary conditions require that

$$0 = R(r_1) = aJ_n(\omega r_1) + bY_n(\omega r_1)\,,$$
$$0 = R(r_2) = aJ_n(\omega r_2) + bY_n(\omega r_2)\,.$$

These equations have a nontrivial solution for a and b if and only if the determinant vanishes:

$$0 = \begin{vmatrix} J_n(\omega r_1) & Y_n(\omega r_1) \\ J_n(\omega r_2) & Y_n(\omega r_2) \end{vmatrix} = J_n(\omega r_1)Y_n(\omega r_2) - Y_n(\omega r_1)J_n(\omega r_2)\,.$$

This equation (which can't be solved by exact methods) determines the allowed eigenfrequencies ω_{nj}. Then either of the boundary conditions can be solved to yield the ratio of a to b in each corresponding eigenfunction. (Alternatively, start by using one of the equations to fix a/b, then use the other one to determine the allowed eigenvalues.)

(b) For a *fixed* n (but varying j) what orthogonality and completeness relations do you expect the functions $R_{nj}(r)$ to obey?

The problem

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R + \omega^2 R = 0\,, \qquad R(r_1) = 0 = R(r_2)\,,$$

is a regular Sturm–Liouville problem, so the eigenfunctions form a complete, orthogonal set. (We should check that we have not missed any modes. A standard integration-by-parts argument shows that ω^2 must indeed be positive. Alternatively, the lowest eigenfunction in our list clearly has no nodes inside the interval — since otherwise we could scale that zero to an endpoint to create a mode with a lower positive eigenvalue — so it is indeed the lowest eigenfunction of all, by the Sturm theory.) The differential equation can be rewritten in the explicit SL form

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{n^2}{r}R = -\omega^2 rR\,,$$

which shows that the weight function is r. Therefore, the orthogonality relation is

$$\int_{r_1}^{r_2} R_{nj}(r)R_{nk}(r)\,r\,dr = 0 \quad \text{unless } j = k\,.$$

(The Bessel functions for different n are *not* orthogonal; that burden is carried by the trig functions in Φ and Ψ.) To get *orthonormal* basis functions we need to divide R_{nj} by

$$\|R_{nj}\| = \sqrt{\int_{r_1}^{r_2} R_{nj}(r)^2\,r\,dr}\,.$$

Therefore, the completeness relation is

$$\sum_{j=1}^{\infty} \frac{1}{\|R_{nj}\|^2} R_{nj}(r)R_{nj}(r') = \frac{1}{r}\delta(r-r')\,.$$

(The r in the denominator could also be written r' or $\sqrt{rr'}$. One way to check that that factor is correct is to multiply the completeness relation by $rR_{nk}(r)/\|R_{nk}\|$ and integrate over r, using the orthonormality relation to get $R_{nk}(r)/\|R_{nk}\|$ back again.)

(c) Show how to expand an arbitrary function $f(r,\theta)$ (defined on the annulus) as a series in the functions Φ_{nj} and Ψ_{nj}. (Now n and j both vary.)

Since the $R_{nj}(r)/\|R_{nj}\|$ are orthonormal and complete, and so are the trig functions when multiplied by $1/\sqrt{\pi}$, we can expand

$$f(r,\theta) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}[a_{nj}\Phi_{nj} + b_{nj}\Psi nj]$$

with

$$a_{nj} = \frac{1}{\pi\|R_{nj}\|^2}\int_0^{2\pi} d\theta \int_{r_1}^{r_2} r\,dr\,\sin(n\theta)R_{nj}(r)\,f(r,\theta)$$

and the analogous formula for b_{nj}. No, that isn't quite right: $n = 0$ creates its usual problems. The modes Φ_{0j} don't exist, and the Ψ_{0j} have an extra $\frac{1}{2}$ in their coefficient formula.

2. *(50 pts.)* Consider Laplace's equation in the region

$$0 \le r < r_1\,, \qquad 0 < \theta < \frac{\pi}{2}\,.$$

(a) Solve the problem with the boundary conditions

$$u(r,0) = 0$$

and either

(regular) $$u\left(r, \frac{\pi}{2}\right) = 0\,, \qquad u(r_1,\theta) = g(\theta)$$

or

(honors) $$u\left(r, \frac{\pi}{2}\right) = f(r)\,, \qquad u(r_1,\theta) = 0\,.$$

Regular: This is a rather standard problem, so I'll just state the result. (But see also the first steps of the honors solution and switch the signs.) In θ we have a Fourier sine series with $L = \frac{\pi}{2}$, so

$$u(r,\theta) = \sum_{n=1}^{\infty} b_n r^{2n} \sin(2n\theta)$$

with

$$b_n = r_1^{-2n}\, \frac{4}{\pi} \int_0^{\pi/2} \sin(2n\theta) g(\theta)\, d\theta\,.$$

Honors: We must expect oscillatory-type solutions in the radial direction and exponential-type solutions in the angular direction, so the sign of the separation constant must be the opposite of that in the previous case:

$$\frac{\Theta''}{\Theta} = +k^2 = -r^2\,\frac{R''}{R} - r\,\frac{R'}{R}\,.$$

From the Dirichlet condition on the bottom edge we see that $\Theta(\theta) \propto \sinh(k\theta)$. The radial solutions are linear combinations of r^{ik} and r^{-ik} , which we can also write e^{iku} and e^{-iku} with $u = \ln r$. The combination vanishing at r_1 is $R(r) = \sin[k(u-u_1)]$, $u_1 = \ln r_1$. So a convenient new variable is $v = u_1 - u$. As $r \to 0$, v approaches $+\infty$ (hence the strange sign in its definition). Therefore, the appropriate eigenfunction expansion is a Fourier sine transform.

$$u(r,\theta) = \int_0^{\infty} B(k) \sinh(k\theta) \sin(kv)\, dv\,,$$

$$B(k)\sinh(\pi k/2) = \frac{2}{\pi} \int_0^{\infty} \sin(kv) f(r)\, dv\,.$$

To finish up we divide by the sinh and either write the r in $f(r)$ as $r = r_1 e^{-v}$, or write

$$\int_0^{\infty} \sin(kv) \cdots\, dv \quad \text{as} \quad \int_0^{r_1} \sin[k(\ln r_1 - \ln r)] \cdots\, \frac{dr}{r}\,.$$

(b) *(essay)* Explain how the results in (a) would be useful in solving the wave equation in that region with time-independent nonhomogeneous boundary conditions.

The solution of the wave equation will be a sum of a solution with the corresponding *homogeneous* boundary conditions and a steady-state solution that satisfies Laplace's equation with the given nonhomogeneous boundary data. If the data are all Dirichlet, the steady-state solution will be a sum of three terms, the two we just found plus something similar to the honors solution to handle the data on the edge $\theta = 0$. The steady-state solution must be subtracted from the initial data for the wave equation before solving the homogenized wave problem.