Math. 412 (Fulling) 11 December 2012

Final Examination – Solutions

Calculators may be used for simple arithmetic operations only!

Some possibly useful information

Laplacian operator in polar coordinates:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,.$$

Laplacian operator in spherical coordinates ("physicists' notation"):

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\,.$$

Spherical harmonics satisfy

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_l^m(\theta,\phi) = -l(l+1)Y_l^m(\theta,\phi).$$

Bessel's equation:

$$\frac{\partial^2 Z}{\partial z^2} + \frac{1}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{n^2}{z^2}\right) Z = 0 \quad \text{has solutions } J_n(z) \text{ and } Y_n(z)\,.$$

$$\frac{\partial^2 Z}{\partial z^2} + \frac{2}{z}\frac{\partial Z}{\partial z} + \left(1 - \frac{l(l+1)}{z^2}\right) Z = 0 \quad \text{has solutions } j_l(z) \text{ and } y_l(z)\,.$$

Legendre's equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0 \quad \text{has a nice solution } P_l(\cos\theta)\,.$$

Famous Green function integrals:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-k^2 t}\, dk = \frac{1}{\sqrt{4\pi t}}\, e^{-x^2/4t}, \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, e^{-|k|y}\, dk = \frac{1}{\pi}\frac{y}{x^2+y^2}\,.$$

Hyperbolic function identities:

$\sinh(A\pm B) = \sinh A\cosh B \pm \cosh A \sinh B, \qquad \cosh(A\pm B) = \cosh A\cosh B \pm \sinh A\sinh B.$

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1. *(30 pts.)* Classify each equation as

   (i) elliptic, hyperbolic, or parabolic,

and

   (ii) linear homogeneous, linear nonhomogeneous, or nonlinear.

   (a) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0$$

Parabolic, linear homogeneous.

   (b) $$\frac{\partial^2 u}{\partial x^2} + (1 + u^2)\frac{\partial^2 u}{\partial y^2} = 0$$

Elliptic, nonlinear.

   (c) $$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \cos(2t) = 0$$

Hyperbolic, linear nonhomogeneous.

2. *(40 pts.)* Solve Laplace's equation in a ball,

$$\nabla^2 u = 0 \quad \text{for} \quad 0 \le r < R, \qquad \frac{\partial u}{\partial r}(R, \theta, \phi) = f(\theta, \phi).$$

   (You may jump right to the answer if you know it. The spherical harmonic notation is strongly advised.)

The general solution of Laplace's equation in spherical coordinates with regularity at the origin is

$$u(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} r^l Y_l^m(\theta, \phi)\,.$$

Impose the boundary condition

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_{lm} l R^{l-1} Y_l^m(\theta, \phi)\,.$$

Since the spherical harmonics are orthonormal, it follows that

$$c_{lm} = l^{-1} R^{1-l} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, Y_l^m(\theta, \phi)^* f(\theta, \phi)\,.$$

3. *(60 pts.)*

(a) Solve by separation of variables or an equivalent transform technique:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad (0 < x < \infty, \quad 0 < t < \infty),$$
$$\frac{\partial u}{\partial x}(0,t) = 0 \quad (0 < t < \infty),$$
$$u(x,0) = f(x) \quad (0 < x < \infty).$$

I will use the transform language since it's briefer. The infinite x interval and Neumann boundary condition suggest a Fourier cosine transform in x:

$$\frac{\partial U}{\partial t} + \omega^2 U = 0\,, \quad U(\omega,0) = F(\omega)\,.$$

So

$$U(\omega,t) = F(\omega) e^{-\omega^2 t}\,.$$

So far I didn't say how the transform is normalized; we need a net factor of $\frac{2}{\pi}$. So if you define

$$F(\omega) = \int_0^\infty \cos(\omega x) f(x)\, dx\,, \tag{1}$$

you have

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \cos(\omega x) e^{-\omega^2 t} F(\omega)\, d\omega\,. \tag{2}$$

(b) Find the Green function that gives the solution to (a) in the form

$$u(x,t) = \int_0^\infty G(x,z,t) f(z)\, dz\,.$$

(There are two methods. **Do** evaluate the integral if your method leads to one.)

Method 1: Substitute (1) into (2) and rearrange:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty dz \int_0^\infty d\omega \cos(\omega x)\cos(\omega z) e^{-\omega^2 t} f(z)\,.$$

So we can identify

$$\begin{aligned}
G(x,z,t) &= \frac{2}{\pi}\int_0^\infty d\omega \cos(\omega x)\cos(\omega z) e^{-\omega^2 t} \\
&= \frac{1}{2\pi}\int_0^\infty d\omega \left[e^{i\omega(x+z)} + e^{-i\omega(x+z)} + e^{i\omega(x-z)} + e^{-i\omega(x-z)}\right] e^{-\omega^2 t} \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty d\omega \left[e^{i\omega(x+z)} + e^{i\omega(x-z)}\right] e^{-\omega^2 t} \\
&= \frac{1}{\sqrt{4\pi t}}\left[e^{-(x+z)^2/4t} + e^{-(x-z)^2/4t}\right].
\end{aligned}$$

Method 2: We know that the first famous integral, with x replaced by $x - z$, is the Green function for the analogous problem on the whole line. By the method of images, the Green function for our problem is obtained by adding the same function of z replaced by $-z$.

4. *(40 pts.)* Solve the heat equation in the region between two concentric spheres:

$$\frac{\partial u}{\partial t} = \nabla^2 u \quad \text{for} \quad 1 \le r < 2\,,$$

$$u(t,1,\theta,\phi) = 0, \quad u(t,2,\theta,\phi) = 0\,,$$

$$u(0,r,\theta,\phi) = f(r,\theta) \quad (\text{independent of } \phi\,).$$

(Note that ϕ is the **azimuthal** angle, not the polar one.)

Separate variables as $u = \Psi\omega(r,\theta,\phi)e^{-\omega^2 t}$, arriving at $-\omega^2\Psi_\omega = \nabla^2\Psi_\omega$, the Laplacian in spherical coordinates being given on the first page of the test. The quickest way to proceed is to notice that since the data function in this problem is independent of ϕ, the relevant eigenfunctions will be, too; therefore, we can discard all the ϕ derivatives and get

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) = -\omega^2 U.$$

Multiply by r^2 and separate variables again: $U = R(r)\Theta(\theta)$,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\Theta = 0$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)R - \omega^2 r^2 R\,.$$

So $\Theta = P_l(\cos\theta)$ and

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} = \left(\frac{l(l_1)}{r^2} - \omega^2\right)R\,,$$

whose solutions are $j_l(\omega r)$ and $y_l(\omega r)$, the spherical Bessel functions. We need to choose a linear combination of them that vanishes at $r = 1$;

$$R_l(\omega,r) \equiv j_l(\omega r)y_l(\omega) - y_l(\omega r)j_l(\omega)$$

will do. We also need

$$0 = R_l(\omega,2) = j_l(2\omega)y_l(\omega) - y_l(2\omega)j_l(\omega)\,;$$

this is the (regular Sturm–Liouville) eigenvalue condition to be solved for the allowed values ω_j. The full solution now is

$$u(t,r,\theta,\phi) = \sum_{j=1}^{\infty}\sum_{l=0}^{\infty} C_{lj}R_l(\omega_j,r)P_l(\cos\theta)e^{-\omega_j{}^2 t}\,.$$

The coefficients are

$$C_{ln} = \frac{\langle U_{lj}, f\rangle}{\|U_{lj}\|^2}\,, \qquad U_{lj}(r,\theta) = R_l(\omega_j,r)P_l(\cos\theta)\,.$$

The numerator and denominator are messy to write out; they both involve integrations of the type

$$\int_1^2 r^2 dr \int_0^\pi \cos\theta\, d\theta\,.$$

There is no need to integrate over ϕ, or even to multiply by 2π because that will cancel. Remember, however, that the Legendre polynomials are not normalized to unit norm.

5. *(Brief essays – 30 pts.)* Imagine that you are grading homework, or working in a help session, in this course. How would you explain to a student how [s]he is going wrong?

   (a) The problem is the wave equation on the real line ($-\infty < x < \infty$). The student has separated variables and arrived at mode functions of the form

$$\big(A_\omega \cos(\omega x) + B_\omega \sin(\omega x)\big)\big(C_\omega \cos(\omega t) + D_\omega \sin(\omega t)\big)\,.$$

The solution must be a *linear* combination of normal modes,

$$A_\omega \cos(\omega x)\cos(\omega t) + B_\omega \sin(\omega x)\cos(\omega t) + C_\omega \cos(\omega x)\sin(\omega t) + D_\omega \sin(\omega x)\sin(\omega t)\,.$$

In the product form, solutions for the coefficients that match the data may not exist and may not be unique.

   (b) The problem is the heat equation on an interval of length π with data

$$u(x,0) = f(x)\,, \quad u(0,t) = T\,, \quad u(\pi,t) = 0\,.$$

The student has separated variables and arrived at the equations

$$X_n''(x) = -n^2 X_n(x)\,, \quad X_n(0) = T\,.$$

You can't impose a nonhomogeneous boundary condition on the individual normal modes! When you add them, the condition will not be satisfied. In this problem you should find a steady-state solution first and subtract it off.

   (c) The problem is Laplace's equation on the semiinfinite strip $0 < x < \infty\,,\ 0 < y < L$ with boundary data

$$u(x,0) = 0\,, \quad u(x,L) = f(x)\,, \quad u(0,y) = 0\,.$$

The student has written down a sum

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}\, x) \sinh(\sqrt{\lambda_n}\, y)\,.$$

What is λ_n here? Because the data interval is infinite, the solution should be a transform (integral), not a sum.