Math. 412 (Fulling) 19 September 2012

Test A – Solutions

Calculators may be used for simple arithmetic operations only!

When a question appears in two versions, answer the version appropriate to your status (honors or regular). Then work on the other version if you have time.

1. *(15 pts.)* Classify each of these equations as linear homogeneous, linear nonhomogeneous, or nonlinear. (In each case, u is the unknown function.)

(a) $u(x) + \int_0^x t^2 u(t)\,dt = \cos x\,.$

linear nonhomogeneous

(b) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 3u\,.$

linear homogeneous

(c) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + u^2\,\dfrac{\partial u}{\partial x}\,.$

nonlinear

2. *(40 pts.)* Consider the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \infty, \quad -\infty < t < \infty),$$

$$u(0,t) = 0 \qquad (-\infty < t < \infty),$$

$$u(x,0) = 0 \qquad (0 < x < \infty),$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \qquad (0 < x < \infty),$$

where

$$g(x) = -2(x-2)e^{-(x-2)^2} \qquad \textbf{(regular)}$$

or

$$g(x) = \begin{cases} 1 & \text{for } 1 < x < 2, \\ 0 & \text{elsewhere.} \end{cases} \qquad \textbf{(honors)}$$

Answer (a) and (b) in whichever order you prefer (or together).

(a) Sketch the solution as a function of x for $t=1$, $t=2$, $t=3$, and $t=4$.

(b) Write down the solution by the d'Alembert (characteristics) method.

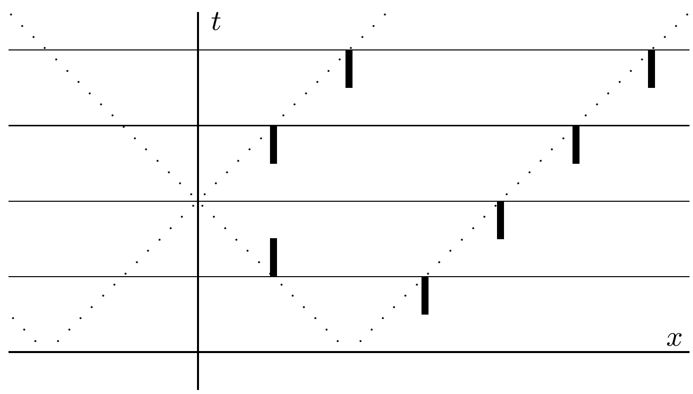
Regular problem: Note that g can easily be integrated:

$$G(x) \equiv e^{-(x-2)^2} \;\Rightarrow\; G' = g\,.$$

So G is a pulse sharply peaked around $x = 2$. To satisfy the boundary condition at $x = 0$ we need to pass to the even extension of G (corresponding to the odd extension of g). The extended G equals the old G for positive x and equals $\exp\left[-(x+2)^2\right]$ for negative x. Then according to d'Alembert's formula,

$$u(x,t) = \tfrac{1}{2}[G(x+t) - G(x-t)].$$

Since the extended G itself consists of 2 pulses, the full solution has 4 pulses, one of which moves to the left from $x = -2$ and hence never enters the physical region. At $t = 2$ the left-moving pulse that originated at $x = 2$ hits the boundary and leaves the physical region, and the right-moving pulse from $x = -2$ enters the physical region; these two pulses have opposite signs, as always for a Dirichlet boundary. The right-moving pulses are negative, the left-moving ones are positive. The right-moving pulse from $x = 2$ stays in the physical region forever. (I will not describe in words what happens for $t < 0$.) At $t = 0$ the pulses cancel each other, in keeping with the initial condition $u(x,0) = 0$. In space-time the pulses move like this:



Honors problem: This time we can take

$$G(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ x - 1 & \text{if } 1 < x < 2, \\ 1 & \text{if } x > 2. \end{cases}$$

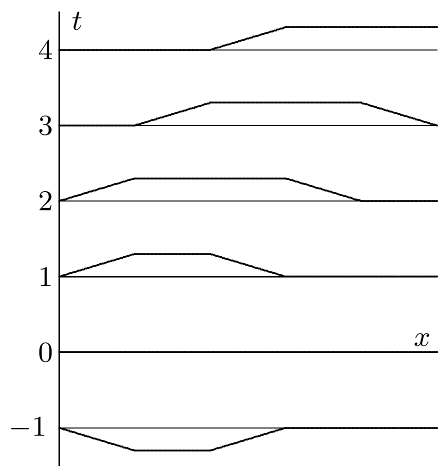
Thus G is not a localized pulse, but more like a moving slab. The even extension has also

$$G(x) = \begin{cases} 1 & \text{if } x < -2, \\ -x - 1 & \text{if } -2 < x < -1, \\ 0 & \text{if } -1 < x < 0. \end{cases}$$

The full solution

$$u(x,t) = \tfrac{1}{2}[G(x+t) - G(x-t)]$$

is a depression between two slabs moving to the left minus a depression between two slabs moving to the right. It helps to graph both terms at time t and then sketch their sum; after some intermediate sketches on separate paper one sees that the correct picture is



Only the physical region of x is shown. I included time $t = -1$ because it shows an incoming pulse that will reflect and invert when it is close to the boundary. It's also instructive to look at small fractional values of t such as 1.5, 0.5, 0.2; see attached *Mathematica* session.

3. *(45 pts.)*

(a) Let $f(x) = x$ on the interval $0 < x < \pi$. Find the Fourier sine series of f on that interval.

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \qquad (L = \pi \text{ here.})$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx = \frac{2}{\pi} \left[-\frac{x}{n} \cos(nx) \Big|_0^{\pi} + \int_0^{\pi} \frac{\cos(nx)}{n}\, dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{x}{n} \cos(nx) + \frac{1}{n^2} \sin(nx) \right]_0^{\pi} = \frac{2}{\pi} \left[-\frac{\pi}{n} \cos(n\pi) \right] = \frac{(-1)^{n-1} 2}{n} .$$

(b)

(1) Does the series converge pointwise?

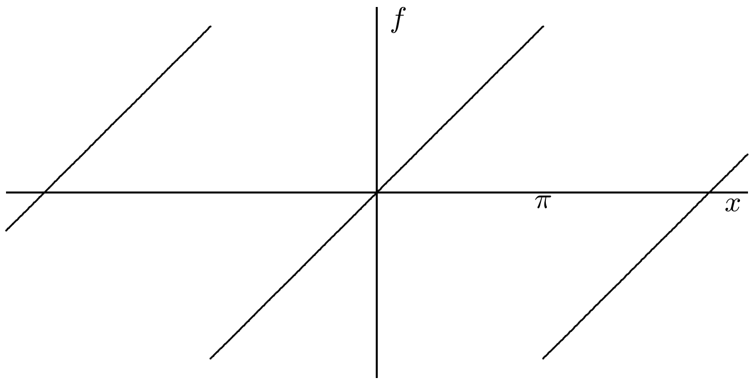
Yes, the function f is piecewise smooth.

(2) Does it converge uniformly?

No, the function (extended to be odd and piecewise periodic) is not continuous at $x = \pi$, for example.

(3) To what does it converge outside the interval?

the odd periodic extension just described:



(c) Solve the wave propagation problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad -\infty < t < \infty),$$

$$u(0,t) = 0 = u(\pi,t) \qquad (-\infty < t < \infty),$$

$$u(x,0) = x, \quad \frac{\partial u}{\partial t}(x,0) = 0 \qquad (0 < x < \pi)$$

by the Fourier method (separation of variables). (Don't go through all the steps of "discovering" the right separated solutions. Just use the results of (a) and (b).)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2}{n} \sin(nx)\cos(nt).$$